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Twisting Somersault*

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Abstract. We give a dynamical system analysis of the twisting somersaults using a reduction to a time-dependent Euler equation for nonrigid body dynamics. The central idea is that after reduction the twisting motion is apparent in a body frame, while the somersaulting (rotation about the fixed angular momentum vector in space) is recovered by a combination of dynamic phase and geometric phase. In the simplest "kick-model" the number of somersaults m and the number of twists n are obtained through a rational rotation number $W = m/n$ of a (rigid) Euler top. Using the full model with shape changes that take a realistic time we then derive the master twisting-somersault formula: an exact formula that relates the airborne time of the diver, the time spent in various stages of the dive, the numbers m and n, the energy in the stages, and the angular momentum by extending a geometric phase formula due to Cabrera [*J. Geom. Phys.*, 57 (2007), pp. 1405–1420]. Numerical simulations for various dives agree perfectly with this formula where realistic parameters are taken from actual observations.

Key words. nonrigid body dynamics, geometric phase, biomechanics, rotation number

AMS subject classifications. 70E55, 70E17, 37J35, 92C10

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1. Introduction.

One of the most beautiful Olympic sports is springboard and platform diving, where a typical dive consists of a number of somersaults and twists performed in a variety of forms. The athlete generates angular momentum at take-off and achieves the desired dive by executing shape changes while airborne. From a mathematical point of view the simpler class of dives are those for which the rotation axis and hence the direction of angular velocity remain constant and only the values of the principal moments of inertia are changed by the shape change, but not the principal axis. This is typical in dives with somersaults in a tight tuck position with minimal moments of inertia. The mathematically much more interesting dives include a shape change that moves the principal axis and hence generates a motion in which the rotation axis is not constant. This is typical in twisting somersaults, the object of this paper. We are using modern tools from dynamical systems, in particular from geometric mechanics, to understand this situation. Our findings apply to coupled rigid body dynamics in general, e.g., in other sports like aerial skiing, or to spacecraft attitude control.

The first correct description of the physics of the twisting somersault was given by Frohlich [5]. Frohlich writes with regards to some publication from the 60s and 70s that "several books

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written by or for coaches discuss somersaulting and twisting, and exhibit varying degrees of insight and/or confusion about the physics processes that occur." A full fledged analysis has been developed by Yeadon in a series of classical papers [17, 18, 19, 20]. Frohlich was the first to point out the importance of shape change for generating rotations even in the absence of angular momentum. Our analysis reveals how exactly a shape change generates a change in rotation in the presence of angular momentum. From a modern point of view this is a question raised in the seminal papers by Shapere and Wilczek [13, 14]: "What is the most efficient way for a body to change its orientation?" Our answer involves the generalization of geometric phase in rigid body dynamics [10] to shape-changing bodies recently obtained in [3].

To be able to apply these ideas in our context we first derive a version of the Euler equation for a shape-changing body. Such equations have been obtained in principle in, e.g., [11, 6, 7, 4]. Our form of the equations is particularly simple, and we derive the explicit form of the important time-dependent terms for a system of coupled rigid bodies. By writing the equations in a particular frame we find beauty and simplicity in the equations of motion (see Theorems 1 and 2). We then take a simple particular system of just two coupled rigid bodies (the "one-armed diver") and show how a twisting somersault can be achieved with this model. An even simpler model is the diver with a rotor analyzed in [2], in which all the stages of the dive can be analytically solved for. In the present paper we use an analytically solvable approximation in which the shape change is instantaneous. In this kick-model the dynamics is described by a piecewise smooth dynamical system and we use an extension of Montgomery's geometric phase formula [10] for the reconstruction; see Theorem 6. A similar extension for a different rotation angle was first obtained by Bates, Cushman, and Savev [1].

Throughout the paper we emphasize the geometric mechanics point of view. Hence the translational and rotational symmetry of the problem is reduced, and thus Euler-type equations are found in a co-moving frame. In this reduced description the amount of somersault (i.e., the amount of rotation about the fixed angular momentum vector in space) is not present. Reconstruction allows us to recover this angle by solving an additional differential equation driven by the solution of the reduced equations. It turns out that for a closed loop in shape space the somersault angle can be recovered by a geometric phase formula due to [3]; see Theorem 9.

The structure of the paper is as follows. In section 2 we derive the equations of motion for a system of coupled rigid bodies that is changing shape. The resulting Euler-type equations are the basis for the following analysis. In section 3 we discuss a simplified kick-model, in which the shape change is impulsive. The kick changes the trajectory and the energy, but not the total angular momentum. In section 4 the full model is analyzed, without the kick assumption. Unlike the previous section, here we have to resort to numerics to compute some of the terms. But we show that using the generalized geometric phase formula due to Cabrera [3] gives an exact description and a beautiful geometric interpretation of the mechanics behind the twisting somersault.

2. Euler equations for coupled rigid bodies.

Let $\mathbf{l}$ be the constant angular momentum vector in a space fixed frame. Rigid body dynamics usually use a body-frame because in that frame the tensor of inertia is constant. The change from one coordinate system to the other is given by a rotation matrix $R = R(t) \in SO(3)$ such that $\mathbf{l} = R\mathbf{L}$. In the body frame the vector $\mathbf{L}$ is described as a moving vector and only its length remains constant since $R \in SO(3)$. The angular velocity $\mathbf{\Omega}$ in the body frame is the vector such that $\mathbf{\Omega} \times \mathbf{v} = R^t \dot{R}\mathbf{v}$ for any vector $\mathbf{v} \in \mathbb{R}^3$. Even though for a system of coupled rigid bodies the tensor of inertia is generally not a constant, a body frame still gives the simplest equations of motion.

Theorem 1. *The equations of motion for a shape-changing body with angular momentum vector $\mathbf{L} \in \mathbb{R}^3$ in a body frame are*

$$\dot{\mathbf{L}} = \mathbf{L} \times \mathbf{\Omega}, \tag{1}$$

where the angular velocity $\mathbf{\Omega} \in \mathbb{R}^3$ is given by

$$\mathbf{\Omega} = I^{-1}(\mathbf{L} - \mathbf{A}), \tag{2}$$

$I = I(t)$ is the tensor of inertia, and $\mathbf{A} = \mathbf{A}(t)$ is a "momentum shift" (or "shape momentum") generated by the shape change.

Proof. The basic assumption is that the shape change is such that the angular momentum is constant. Let $\mathbf{l}$ be the vector of angular momentum in the space fixed frame; then $\mathbf{l} = R\mathbf{L}$. Taking the time derivative gives $\mathbf{0} = \dot{R}\mathbf{L} + R\dot{\mathbf{L}}$ and hence $\dot{\mathbf{L}} = -R^t\dot{R}\mathbf{L} = -\mathbf{\Omega} \times \mathbf{L} = \mathbf{L} \times \mathbf{\Omega}$. The interesting dynamics is all hidden in the relation between $\mathbf{\Omega}$ and $\mathbf{L}$.

Let $\mathbf{q} = R\mathbf{Q}$, where $\mathbf{Q}$ is the position of a point in the body B in the body frame, and $\mathbf{q}$ is the corresponding point in the space fixed frame. The relation between $\mathbf{L}$ and $\mathbf{\Omega}$ is obtained from the definition of angular momentum which is $\mathbf{q} \times \dot{\mathbf{q}}$ integrated over the body B. The relation $\mathbf{q} = R\mathbf{Q}$ is for a rigid body; for a deforming body we label each point by $\mathbf{Q}$ in the body frame but allow for an additional shape change S, so that $\mathbf{q} = RS\mathbf{Q}$. We assume that $S : \mathbb{R}^3 \to \mathbb{R}^3$ is volume preserving, which means that the determinant of the Jacobian matrix of S is 1. The deformation S need not be linear, but we assume that we are in a frame in which the center of mass is fixed at the origin, so that S fixes that point. Now

$$\begin{aligned} \dot{\mathbf{q}} &= \dot{R}S\mathbf{Q} + R\dot{S}\mathbf{Q} + RS\dot{\mathbf{Q}} = RR^t\dot{R}S\mathbf{Q} + R\dot{S}\mathbf{Q} = R(\mathbf{\Omega} \times S\mathbf{Q}) + R\dot{S}S^{-1}S\mathbf{Q} \\ &= R(\mathbf{\Omega} \times \tilde{\mathbf{Q}} + \dot{S}S^{-1}\tilde{\mathbf{Q}}), \end{aligned} \tag{3}$$

where $\tilde{\mathbf{Q}} = S\mathbf{Q}$. Thus we have

$$\begin{aligned} \mathbf{q} \times \dot{\mathbf{q}} &= R\tilde{\mathbf{Q}} \times R(\mathbf{\Omega} \times \tilde{\mathbf{Q}} + \dot{S}S^{-1}\tilde{\mathbf{Q}}) \\ &= R(|\tilde{\mathbf{Q}}|^2\mathbb{1} - \tilde{\mathbf{Q}}\tilde{\mathbf{Q}}^t)\mathbf{\Omega} + R(\tilde{\mathbf{Q}} \times \dot{S}S^{-1}\tilde{\mathbf{Q}})\,. \end{aligned} \tag{4}$$

Now $\mathbf{l}$ is defined by integrating over the deformed body $\tilde{B}$ with density $\rho = \rho(\tilde{\mathbf{Q}})$, so that $\mathbf{l} = \int \rho\mathbf{q} \times \dot{\mathbf{q}}\, \mathrm{d}\tilde{\mathbf{Q}}$ and using $\mathbf{l} = R\mathbf{L}$ gives

$$\mathbf{L} = \int_{\tilde{B}} \rho(|\tilde{\mathbf{Q}}|^2\mathbb{1} - \tilde{\mathbf{Q}}\tilde{\mathbf{Q}}^t)\, \mathrm{d}\tilde{\mathbf{Q}}\ \mathbf{\Omega} + \int_{\tilde{B}} \rho\tilde{\mathbf{Q}} \times \dot{S}S^{-1}\tilde{\mathbf{Q}}\, \mathrm{d}\tilde{\mathbf{Q}}. \tag{5}$$

The first term is the tensor of inertia I of the shape changed body, and the constant term defines the shape momentum $\mathbf{A}$ so that

$$\mathbf{L} = I\mathbf{\Omega} + \mathbf{A}, \tag{6}$$

as claimed. ∎

Remark 1.1. Explicit formulas for I and $\mathbf{A}$ in the case of a system of coupled rigid bodies are given in the next theorem. When I is constant and $\mathbf{A} = \mathbf{0}$ the equations reduce to the classical Euler equations for a rigid body.

Remark 1.2. For arbitrary time dependence of I and $\mathbf{A}$ the total angular momentum $|\mathbf{L}|$ is conserved; in fact, it is a Casimir of the Poisson structure $\{f, g\} = \nabla f \cdot \mathbf{L} \times \nabla g$.

Remark 1.3. The equations are Hamiltonian with respect to this Poisson structure with Hamiltonian $H = \frac{1}{2}(\mathbf{L} - \mathbf{A})I^{-1}(\mathbf{L} - \mathbf{A})$ such that $\boldsymbol{\Omega} = \partial H/\partial \mathbf{L}$.

For a system of coupled rigid bodies the shape change S is given by rotations of the individual segments relative to some reference segment, typically the trunk. The orientation of the reference segment is given by the rotation matrix R so that $\mathbf{l} = R\mathbf{L}$. The system of rigid bodies is described by a tree that describes the connectivity of the bodies; see the thesis of Tong [15] for the details.

Denote by $\mathbf{C}$ the overall center of mass, and by $\mathbf{C}_i$ the position of the center of mass of body B_i relative to $\mathbf{C}$. Each body's mass is denoted by m_i, and its orientation by R_{α_i}, where α_i denotes the set of angles necessary to describe its relative orientation (e.g., a single angle for a pin joint, or three angles for a ball and socket joint). All orientations are measured relative to the reference segment, so that the orientation of B_i in the space fixed frame is given by RR_{α_i}. The angular velocity $\boldsymbol{\Omega}_{\alpha_i}$ is the relative angular velocity corresponding to R_{α_i}, so that the angular velocity of B_i in the space fixed frame is $R^t_{\alpha_i}\boldsymbol{\Omega} + \boldsymbol{\Omega}_{\alpha_i}$. Finally, I_i is the tensor of inertia of B_i in a local frame with center at $\mathbf{C}_i$ and coordinate axes aligned with the principle axes of inertia. With this notation we have the following.

Theorem 2. *For a system of coupled rigid bodies we have*

$$I = \sum R_{\alpha_i} I_i R^t_{\alpha_i} + m_i(|\mathbf{C}_i|^2 \mathbb{1} - \mathbf{C}_i \mathbf{C}^t_i) \tag{7}$$

and

$$\mathbf{A} = \sum (m_i \mathbf{C}_i \times \dot{\mathbf{C}}_i + R_{\alpha_i} I_i \boldsymbol{\Omega}_{\alpha_i}), \tag{8}$$

where m_i is the mass, $\mathbf{C}_i$ the position of the center of mass, R_{α_i} the relative orientation, $\boldsymbol{\Omega}_{\alpha_i}$ the relative angular velocity such that $R^t_{\alpha_i}\dot{R}_{\alpha_i}\mathbf{v} = \boldsymbol{\Omega}_{\alpha_i}\mathbf{v}$ for all $\mathbf{v} \in \mathbb{R}^3$, and I_i the tensor of inertia of body B_i. The sum is over all bodies B_i including the reference segment, for which the rotation is simply given by $\mathbb{1}$.

Proof. The basic transformation law for body B_i in the tree of coupled rigid bodies is $\mathbf{q}_i = RR_{\alpha_i}(\mathbf{C}_i + \mathbf{Q}_i)$. Repeating the calculation in the proof of Theorem 1 with this particular S and summing over the bodies gives the result. We will skip the derivation of $\mathbf{C}_i$ in terms of the shape change and the geometry of the model and refer the reader to [15] for the details. ■

Remark 2.1. In the formula for I the first term is the moment of inertia of the segment transformed to the frame of the reference segment, while the second term comes from the parallel axis theorem (see, e.g., [8]), applied to the center of mass of the segment relative to the overall center of mass.

Remark 2.2. In the formula for $\mathbf{A}$ the first term is the internal angular momentum generated by the change of the relative center of mass, while the second term originates from the relative angular velocity.

Remark 2.3. When there is no shape change, then $\dot{\mathbf{C}}_i = \mathbf{0}$ and $\mathbf{\Omega}_i = \mathbf{0}$, and hence $\mathbf{A} = \mathbf{0}$.

Remark 2.4. The vectors $\mathbf{C}_i$, $\dot{\mathbf{C}}_i$, and $\mathbf{\Omega}_{\alpha_i}$ are determined by the set of time-dependent matrices $\{R_{\alpha_i}\}$ (the time-dependent "shape") and the joint positions of the coupled rigid bodies (the time-independent "geometry" of the model); see [15] for the details. In particular, also $\sum m_i \mathbf{C}_i = \mathbf{0}$.

In order to describe and numerically compute the rotation matrix R that determines the position of the body in space, we use quaternions. This is convenient because unlike Euler angles the description in quaternions is free of singularities. Specifically, we write $R\mathbf{x} = qx\bar{q}$, where the quaternion q is $q = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}$ and the vector $\mathbf{x}$ on the left-hand side and the pure quaternion $x = x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k}$ on the right-hand side; see, e.g., [15] for more details.

3. A simple model for twisting somersault. Instead of the full complexity of a realistic coupled rigid body model for the human body, e.g., with 11 segments [16] or more, here we are going to show that even when all but one arm is kept fixed it is still possible to do a twisting somersault. The formulas we derive are completely general, so that more complicated shape changes can be studied in the same framework. But in order to explain the essential ingredients of the twisting somersault we choose to discuss a simple example. A typical dive consists of a number of phases or stages in which the body shape is either fixed or not. Again, it is not necessary to make this distinction; the equations of motion are general, and one could study dives where the shape is changing throughout. However, the assumption of rigid body motions for certain times is satisfied to a good approximation in reality and makes the analysis simpler and more explicit. The stages where shape change occurs are relatively short, and considerable time is spent in rotation with a fixed shape. This observation motivates our first approximate model, in which the shape changes are assumed to be impulsive. Hence we have instantaneous transitions between solution curves of rigid bodies with different tensors of inertia and different energy, but the same angular momentum. A simple twisting somersault hence looks like this: The motion starts out as a steady rotation about a principal axis resulting in pure somersault (stage 1), and typically this is about the axis of the middle principle moment of inertia which has unstable equilibrium. After some time a shape change occurs; in our case one arm goes down (stage 2). This makes the body asymmetric and generates some tilt between the new principal axis and the constant angular momentum vector. As a result the body starts twisting with constant shape (stage 3) until another shape change (stage 4) stops the twist, for which the body then resumes pure somersaulting motion (stage 5) until head first entry in the water. The amount of time spent in each of the five stages is denoted by τ_i, where $i = 1, \ldots, 5$. We also use the subscripts $\mathfrak{s}$ for the somersaulting stages 1 and 5, and the subscript $\mathfrak{t}$ for the twisting phase 3.

The energy for pure somersault in stages 1 and 5 is $E_\mathfrak{s} = \frac{1}{2}\mathbf{L}_\mathfrak{s} I_\mathfrak{s}^{-1}\mathbf{L}_\mathfrak{s} = \frac{1}{2}l^2 I_{\mathfrak{s},yy}$. In the kick-model, stages 2 and 4 do not take up any time, so $\tau_2 = \tau_4 = 0$, but they do change the energy and the tensor of inertia. On the momentum sphere $|\mathbf{L}|^2 = l^2$ the dive thus appears like this; see Figure 1: For some time the trajectory of $\mathbf{L}$ on the $\mathbf{L}$-sphere remains at the equilibrium

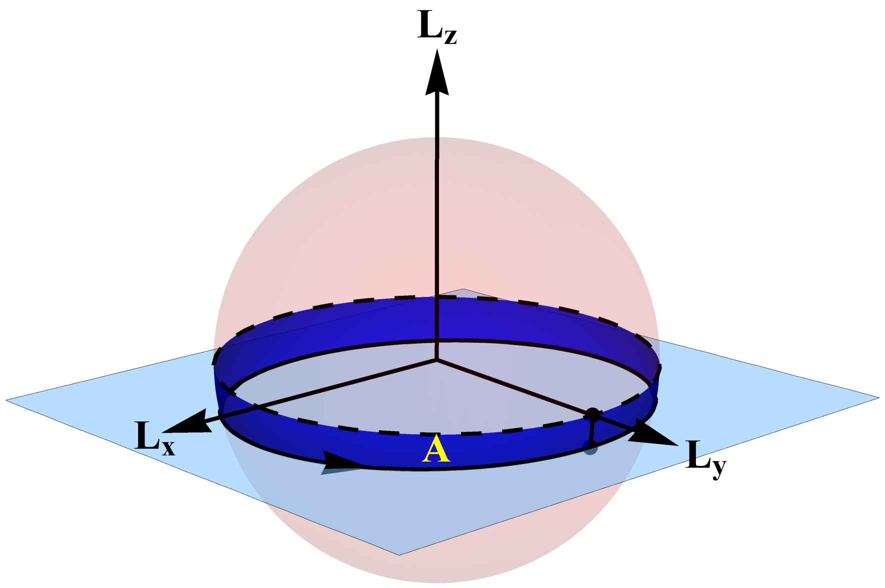


Figure 1. *Twisting somersault on the sphere $|\mathbf{L}| = l$ where the shape change is a kick. The region A bounded by the stage 3 orbit of $\mathbf{L}$ and the equator (dashed) is shaded in dark blue.*

point $L_x = L_z = 0$; then it is kicked into a periodic orbit with larger energy describing the twisting motion. Depending on the total available time a number of full (or half) revolutions are done on this orbit, until another kick brings the solution back to the unstable equilibrium point where it started (or on the opposite end with negative L_y). Finally, some time is spent in the pure somersaulting motion before the athlete completes the dive with head first entry into the water.

The description in the body frame on the sphere $|\mathbf{L}|^2 = l^2$ does not directly contain the somersault rotation about $\mathbf{l}$ in physical space. This is measured by the angle of rotation ϕ about the fixed angular momentum vector $\mathbf{l}$ in space, which is the angle reduced by the symmetry reduction that lead to the description in the body frame, and the angle ϕ will have to be recovered from its dynamic phase and geometric phase. What is visible on the $\mathbf{L}$-sphere is the dynamics of the twisting motion, which is the rotation about the (approximately) head-to-toe body axis. When the shape is constant all motions on the $\mathbf{L}$-sphere except for the separatrices are periodic, but they are not periodic in physical space because ϕ in general has a different period. This is the typical situation in a system after symmetry reduction.

To build a successful dive, a half-integer number of somersaults is required with either a half-integer or integer number of twists. This is achieved by initiating and terminating the orbit on the $\mathbf{L}$-sphere at one of the equilibrium points $(0, \pm l, 0)$. Connecting different equilibrium points corresponds to half-integer twists, while connecting the same equilibrium points corresponds to an integer number of twists. Starting and finishing at these equilibrium points also guarantees that the athlete's take-off and entry into the water is in pure somersaulting motion. As the take-off is performed in the upright position with head-first entry, the change in ϕ, i.e., the amount of rotation about the axis $\mathbf{l}$, has to be a half-integer. So, if necessary,

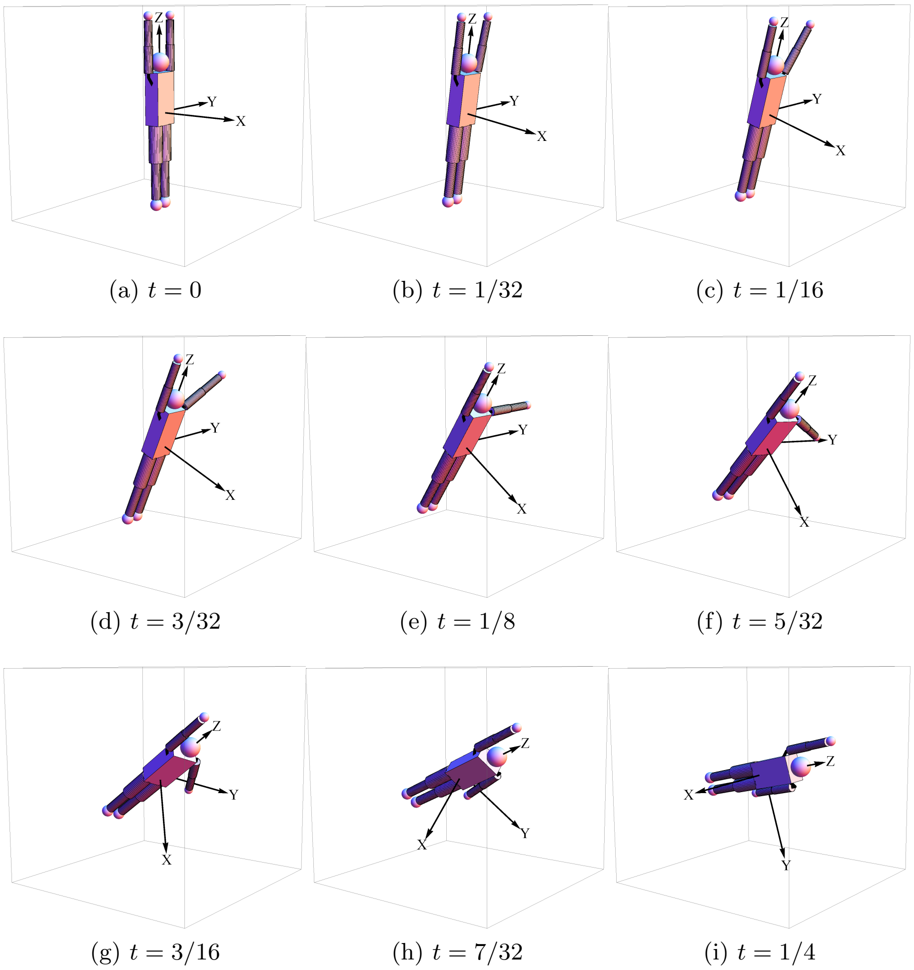
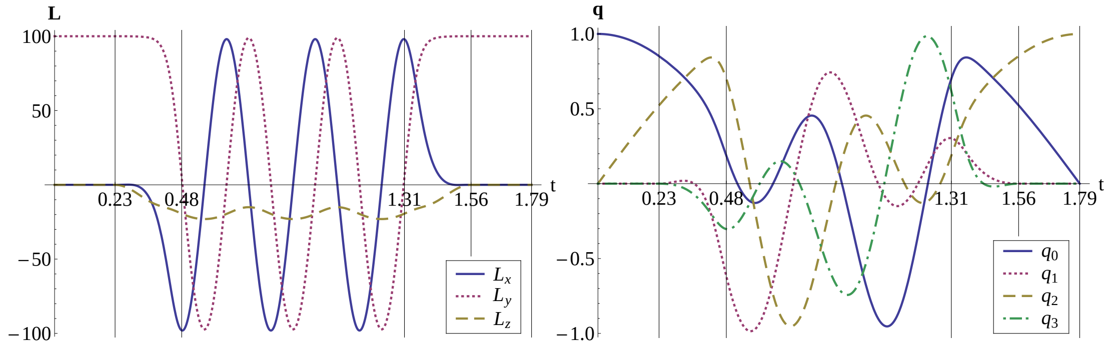


Figure 2. *The arm motion for the twisting somersault. At $t = 0$ the space fixed frame and the body fixed frame are aligned. See the supplementary material* (M105509_01.mp4 [local/web 596KB]) *for a movie of the corresponding twisting somersault.*

the angle ϕ will evolve (without generating additional twist) in the pure somersaulting stages 1 and 5 to meet this criterion.

The orbit for the twisting stage 3 is obtained as the intersection of the angular momentum sphere $|\mathbf{L}|^2 = l^2$ and the Hamiltonian $H = \frac{1}{2}\mathbf{L}I_{\mathfrak{t}}^{-1}\mathbf{L} = E_{\mathfrak{t}}$, where $I_{\mathfrak{t}}$ denotes the tensor of inertia for the twisting stage 3 which in general is nondiagonal (the subscript $\mathfrak{t}$ stands for "twist"). The period of this motion depends on $E_{\mathfrak{t}}$ and can be computed in terms of complete elliptic integrals of the first kind; see below.

Before we proceed with a more detailed analysis of the kick-model we present the numerical solution for a realistic model in which the shape change of Figure 2 takes 0.25 seconds. The solution consists of the five stages described above, and it is presented in two ways: In Figure 3 time series of the components of the vector $\mathbf{L}$ and the components of the quaternion q describing the rotation matrix R are shown. An animation of this dive can be found in



(a) **L** for all stages. (b) q for all stages.

Figure 3. *Twisting somersault with $m = 1.5$ somersaults and $n = 3$ twists. The left pane shows the angular momentum $\mathbf{L}(t)$, and the right pane shows the quaternion $q(t)$ that determines the orientation R. The stages are separated by the vertical thin lines, $\tau_1 = \tau_5 = 0.23$, $\tau_2 = \tau_4 = 1/4$, $\tau_3 = 0.83$. The same trajectory on the **L**-sphere is shown in Figure 4.*

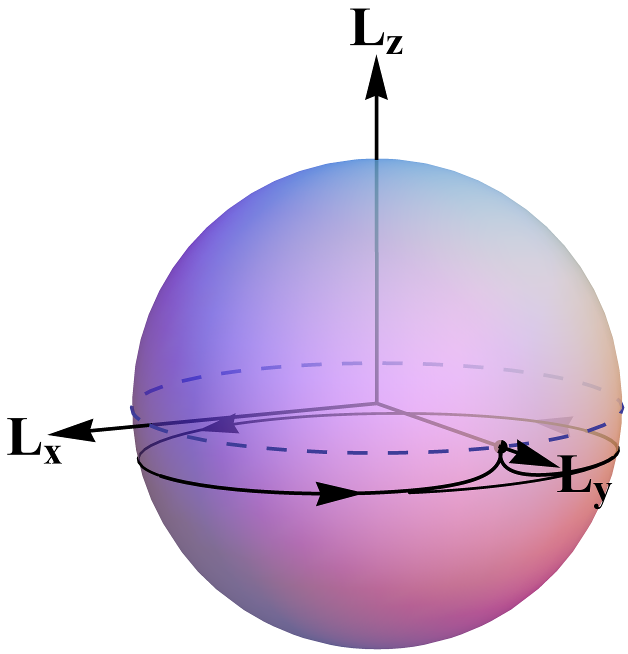


Figure 4. *Twisting somersault with $m = 1.5$ somersault and $n = 3$ twists on the sphere $|\mathbf{L}| = l$. The orbit starts and finishes on the L_y-axis with stages 1 and 5. Shape-changing stages 2 and 4 are the curved orbit segments that start and finish at this point. The twisting somersault stage 3 appears as a slightly deformed circle below the equator (dashed).*

the supplementary material (M105509_01.mp4 [local/web 596KB]). In Figure 4 the trajectory $\mathbf{L}(t)$ is shown as a curve on the sphere $|\mathbf{L}| = l =$ const. Comparison of Figures 4 and 1 shows that qualitatively the kick-model agrees well with the full model. One of the main themes of the following analysis is to extract the essential information about the orientation matrix R contained in the right pane of Figure 3 from some properties of Figure 4, and the theoretical tool to do so will be the geometric phase; see below. But before we resolve this problem we now analyze the kick-model in more detail.

Lemma 3. *In the kick-model the instantaneous shape change of the arm from up ($\alpha = \pi$) to down ($\alpha = 0$) rotates the angular momentum vector in the body frame from $\mathbf{L}_{\mathfrak{s}} = (0, l, 0)^t$ to $\mathbf{L}_{\mathfrak{t}} = R_x(-\chi)\mathbf{L}_{\mathfrak{s}}$, where $R_x(-\chi)$ is a rotation about the x-axis by $-\chi$ and the tilt angle χ is*

$$\chi = \int_0^\pi I^{-1}_{\mathfrak{t},xx}(\alpha) A_x(\alpha) \mathrm{d}\alpha. \tag{9}$$

Here $I^{-1}_{\mathfrak{t},xx}(\alpha)$ is the xx component of the (changing) inverse moment of inertia tensor $I_{\mathfrak{t}}^{-1}$, and the shape momentum is $\mathbf{A} = (A_x(\alpha)\dot\alpha, 0, 0)^t$. In particular the energy after the kick is $E_{\mathfrak{t}} = \frac{1}{2}\mathbf{L}_{\mathfrak{s}} R_x(\chi) I_{\mathfrak{t}} R_x(-\chi)\mathbf{L}_{\mathfrak{s}}$.

Proof. Denote by α the angle of the arm relative to the trunk, where $\alpha = 0$ is arm down and $\alpha = \pi$ is arm up. Let the shape change be determined by $\alpha(t)$, where $\alpha(0) = \pi$ and $\alpha(\tau_2) = 0$. Now $\mathbf{A}$ is proportional to $\dot\alpha$ and hence $\mathbf{A}$ diverges when the duration of the kick τ_2 goes to zero. This means in the limit $\tau_2 \to 0$ we have $|\mathbf{A}| \to \infty$ while $\mathbf{L}$ has constant length. So we replace $\boldsymbol{\Omega} = I^{-1}(\mathbf{L} - \mathbf{A})$ by the approximation $\boldsymbol{\Omega} \approx -I^{-1}\mathbf{A}$. Thus the equations of motion during an infinitesimal kick become

$$\dot{\mathbf{L}} = I^{-1}\mathbf{A} \times \mathbf{L}. \tag{10}$$

Notice that these approximate equations are linear in $\mathbf{L}$. Denote the moving arm as the body B_2 with index 2, and the trunk and all the other fixed segments as a combined body with index 1. Since the arm is moved in the yz-plane, we have $R_{\alpha_2} = R_x(\alpha(t))$ and $\boldsymbol{\Omega}_{\alpha_2}$ parallel to the x-axis. Moreover, the overall center of mass will be in the yz-plane, so that $\mathbf{C}_i$ and $\dot{\mathbf{C}}_i$, $i = 1, 2$, are also in the yz-plane. So we have $\mathbf{C}_i \times \dot{\mathbf{C}}_i$ parallel to the x-axis as well, and hence $\mathbf{A} = (A_x\dot\alpha, 0, 0)^t$. The parallel axis theorem gives nonzero off-diagonal entries only in the yz-component of I, and similarly for $R_{\alpha_2} I_{\mathfrak{s}} R^t_{\alpha_2}$; hence $I_{xy} = I_{xz} = 0$ for this shape change. Thus $I^{-1}\mathbf{A} = (I^{-1}_{\mathfrak{t},xx} A_x \dot\alpha, 0, 0)^t$, and the equation for $\dot{\mathbf{L}}$ can be written as $\dot{\mathbf{L}} = f(t) M \mathbf{L}$, where M is a constant matrix given by $M = \frac{\mathrm{d}}{\mathrm{d}t} R_x(t)|_{t=0}$ and $f(t) = I^{-1}_{\mathfrak{t},xx}(\alpha(t)) A_x(\alpha(t))\dot\alpha$. Since M is constant, we can solve the time-dependent linear equation and find

$$\mathbf{L}_{\mathfrak{t}} = R_x(-\chi)\mathbf{L}_{\mathfrak{s}}, \tag{11}$$

where

$$\chi = \int_0^{\tau_2} I^{-1}_{\mathfrak{t},xx} A_x \dot\alpha \,\mathrm{d}t = \int_0^\pi I^{-1}_{\mathfrak{t},xx}(\alpha) A_x(\alpha)\,\mathrm{d}\alpha\,. \tag{12}$$

We take the limit $\tau_2 \to 0$ and obtain the effect of the kick, which is a change in $\mathbf{L}_{\mathfrak{s}}$ by a rotation of $-\chi$ about the x-axis. The larger the value of χ, the higher the twisting orbit is on the sphere and thus the shorter the period. The energy after the kick is easily found by evaluating the Hamiltonian at the new point $\mathbf{L}_{\mathfrak{t}}$ on the sphere with the new tensor of inertia $I_{\mathfrak{t}}$. ∎

The tensor of inertia after the shape change is denoted by $I_{\mathfrak{t}}$. It does not change by much in comparison to $I_{\mathfrak{s}}$ but is now nondiagonal; however, with the rotation $R_x(\mathcal{P})$ it can be rediagonalized to

$$J = \operatorname{diag}(J_x, J_y, J_z) = R_x(-\mathcal{P}) I_{\mathfrak{t}} R_x(\mathcal{P}), \tag{13}$$

where in general the eigenvalues are distinct. The precise formula for $\mathcal{P}$ depends on the inertia properties of the model; the value for a realistic model with 10 segments can be found in [15]. Formulas for $A_x(\alpha)$ in terms of realistic inertial parameters are also given in [15].

In stage 3 the twisting somersault occurs, where we assume an integer number of twists occur. This corresponds to an integer number of repetitions of the periodic orbit of the rigid body with energy $E_{\mathfrak{t}}$ and tensor of inertia $I_{\mathfrak{t}}$. Let $T_{\mathfrak{t}}$ be the period of this motion. As already pointed out the amount of rotation about the fixed angular momentum vector $\mathbf{l}$ cannot be directly seen on the $\mathbf{L}$-sphere. Denote this angle of rotation by ϕ. We need the total change $\Delta\phi$ to be an odd multiple of π for head-first entry. The amount $\Delta\phi$ can be split into a dynamic phase and a geometric phase, where the geometric phase is given by the solid angle S enclosed by the curve on the $\mathbf{L}$-sphere [10, 1].

We are going to derive the formula for $\Delta\phi$ here because we need a special normalization appropriate for our setting. The essential ingredient in this formula is the solid angle S enclosed by the orbit on the $\mathbf{L}$-sphere; see [10, 9, 1]. In the following two lemmas we consider the rigid body with constant diagonal moments of inertia J and energy E. When we return to the twisting somersault this will be applied to the twisting phase where $E = E_{\mathfrak{t}}$.

Lemma 4. *The solid angle on the sphere $\mathbf{L}^2 = l^2$ enclosed by the intersection with the ellipsoid $\mathbf{L}J^{-1}\mathbf{L} = 2E$ is given by*

$$S(h,\rho) = 2\pi - \frac{4hg}{\pi}\Big(\Pi(\nu, k^2) - K(k^2)\Big), \tag{14}$$

where $k^2 = \rho(1-h\rho)/(2h+\rho)$, $\nu = 1 - h\rho$, $g = (1 + h/\rho)^{-1/2}$, $h = (EJ_y/l^2 - 1/2)/\mu$, $\rho^2 = (1 - J_y/J_z)/(J_y/J_x - 1)$, $\mu^2 = (J_y/J_x - 1)(1 - J_y/J_z)$.

Proof. We do a scaling similar to that done in [12]. Introducing $L_x = lx$, $L_y = ly$, $L_z = lz$ the equation $\frac{1}{2}\mathbf{L}J^{-1}\mathbf{L} = E$ can be written in dimensionless form $(J_y/J_x)x^2 + y^2 + (J_y/J_z)z^2 = 2EJ_y/l^2$. Cylindrical coordinates for the scaled sphere are $x = \sqrt{1-z^2}\sin\psi$, $y = \sqrt{1-z^2}\cos\psi$, and they introduce canonical coordinates (z,ψ). The critical point $z = 0$, $\psi = 0$ is a saddle point of the Hamiltonian, and the energy is shifted so that the critical value of Hamiltonian is zero. Moreover, time is scaled by the eigenvalue μ of the linearized Hamilton equations. Thus we arrive at

$$H(z,\psi) = \frac{1}{2}\left(z^2(\rho + \rho^{-1}\sin^2\psi) - \rho^{-1}\sin^2\psi\right).$$

The solid angle is determined by the area between the curve $H(z,\psi) = h$ and the equator $z = 0$ of the sphere. The reason for this is that the projection of the (normalized) sphere onto the cylinder (z,ψ) is area preserving. Thus we find $S = \int_0^{2\pi} z \, \mathrm{d}\psi$, where z is obtained by taking the positive root of $H(z,\psi)=h$. We are only interested in solutions where z is defined for all ψ, which corresponds to twisting motion where $0<2h\le\rho$. Thus we find that

$$S(h,\rho)=\int_0^{2\pi}\frac{\sqrt{2h\rho+\sin^2\psi}}{\sqrt{\rho^2+\sin^2\psi}}\,\mathrm{d}\psi \tag{15}$$

and this is a complete elliptic integral that can be written in the stated form using standard techniques. ■

Remark 4.1. The essential parameter is EJ_y/l^2 inside h, and all other dependencies are on certain dimensionless combinations of moments of inertia.

Remark 4.2. We reiterate that in our normalization, S is the solid angle enclosed by the curve and the equator. When $h\to\rho/2$, equation (15) gives 2π as desired. Similarly, when $h\to 0$, the minimal area $4\arctan\rho^{-1}$ is attained, which goes to zero in the symmetric case $J_x=J_y>J_z$ since $\rho\to\infty$.

Using this lemma we can find simple expressions for the period and rotation number by noticing that the action variables of the integrable system on the **L**-sphere are proportional to the product of S and l.

Lemma 5. *The derivative of the action $\mathcal{I}=lS/(2\pi)$ with respect to the energy E gives the inverse of the frequency $(2\pi)/T$ of the motion, such that the period T is (notation as in Lemma 4)*

$$T=\frac{4\pi g}{\mu l}K(k^2)\,, \tag{16}$$

and the derivative of the action $\mathcal{I}$ with respect to l gives the rotation number as $-2\pi W=S-2ET/l$, and hence a splitting into geometric phase and dynamic phase.

Proof. The main observation is that the symplectic form on the **L**-sphere of radius l is the area-form on the sphere divided by l and that the solid angle on the sphere of radius l is the enclosed area divided by l^2. Note that unlike in Lemma 4 we are not using scaled variables. Thus the action of the integrable reduced Euler equations denoted by $\mathcal{I}$ (in the twisting stage with constant moment of inertia I and $\mathbf{A}=\mathbf{0}$) is $\mathcal{I}=lS/(2\pi)$ and the area is l^2S. Clearly the action of the integrable system is determined by the solid angle, up to a factor. The reason that the essential object is the solid angle is that the Euler top has scaling symmetry: If **L** is replaced by $s\mathbf{L}$, then E is replaced by s^2E and the solid angle remains the same. The scaling symmetry implies that the essential parameter is the ratio E/l^2 which is invariant under scaling. So the solid angle is a function of E/l^2 only, as we have shown in the previous lemma. The formulas claimed in this lemma follow from simple implicit differentiation. When a one degree of freedom system is written in terms of action-angle variables, we have $E=H(\mathcal{I})$, and the frequency of motion is given by $\partial H/\partial\mathcal{I}$; hence the period is $T/(2\pi)=\partial\mathcal{I}/\partial E$. Therefore differentiating the action $\mathcal{I}=lS(E/l^2)/(2\pi)$ with respect to E gives

$$\frac{T}{2\pi}=\frac{\partial\mathcal{I}}{\partial E}=\frac{\partial lS(E/l^2)}{2\pi\partial E}=\frac{1}{2\pi l}S'(E/l^2) \tag{17}$$

and standard identities of complete elliptic integrals give the stated result. When a two degree of freedom integrable system is written in terms of action-angle variables, we have $E = H(\mathcal{I}_1, \mathcal{I}_2)$. Now the frequencies are given by $\partial H/\partial \mathcal{I}_i$, and for given E their ratio defines the rotation number W. By implicit differentiation at constant energy a formula for W can be derived. In the present case we have $\mathcal{I}_1 = \mathcal{I}$ and $\mathcal{I}_2 = l$. Specifically, l is the momentum of a global S^1 action and hence independent of E. Thus the simple formula $W = -\partial\mathcal{I}/\partial l$ results. Using the form of $\mathcal{I}$ as above then gives

$$-2\pi W = -2\pi\frac{\partial \mathcal{I}}{\partial l} = \frac{\partial l S(E/l^2)}{\partial l} = S(E/l^2) - \frac{2E}{l^2}S'(E/l^2) = S - \frac{2ET}{l}. \tag{18}$$

■

Remark 5.1. What is not apparent in these formulas is that the scaled period lT is a relatively simple complete elliptic integral of the first kind (depending on E/l^2 only), while S and W are both complete elliptic integrals of the third kind (again depending on E/l^2 only).

Remark 5.2. The splitting of the rotation number into geometric phase and dynamic phase for the Euler top goes back to Montgomery [10]. His formula gives the rotation number modulo 1 only. By choosing a particular set of coordinates for the reduction, Bates, Cushman, and Savev [1] were able to remove the modulo operation in the formula. In Lemma 5 we have derived a related formula without the modulo, but it turns out to be different. The explanation is that the rotation number in an integrable system is only defined up to modular transformations. This results from the fact that actions are only defined up to transformations from the special linear group over the integers. Specifically, our "solid angle" is different in that it is measured relative to the equator. Accordingly our rotation number differs from the rotation number in [1].

This concludes our two lemmas about the dynamics of the rigid body, which we are now going to apply specifically to the twisting stage 3. The main observation is that the rotation numbers times 2π gives the amount the somersault angle has changed during one period of the twisting motion.

Theorem 6. *The total amount of rotation $\Delta\phi_{kick}$ about the fixed angular momentum axis* $\mathbf{l}$ *for the kick-model when performing n twists is given by*

$$\Delta\phi_{kick} = (\tau_1 + \tau_5)\frac{2E_\mathfrak{s}}{l} + \tau_3\frac{2E_\mathfrak{t}}{l} - nS\,. \tag{19}$$

The first terms are the dynamic phase where $E_\mathfrak{s}$ is the energy in the somersault stages and $E_\mathfrak{t}$ is the energy in the twisting somersault stage. The last term is the geometric phase where S is the solid angle enclosed by the orbit in the twisting somersault stage. For equal moments of inertia $J_x = J_y$ the solid angle S is

$$S = 2\pi\sin(\chi + \mathcal{P}) \tag{20}$$

and in general is given by Lemma 4 where $E = E_\mathfrak{t}$. To perform n twists the time necessary is $\tau_3 = nT_\mathfrak{t}$, where for equal moments of inertia $J_x = J_y$ the period $T_\mathfrak{t}$ is

$$T_\mathfrak{t} = \frac{2\pi}{l}\frac{(J_y^{-1} - J_z^{-1})^{-1}}{\sin(\chi + \mathcal{P})} \tag{21}$$

and in general is given by Lemma 5 where $T = T_\mathfrak{t}$.

Proof. Each stage contributes to the somersault angle (as long as $\tau_i > 0$). For the pure somersaulting stages 1 and 5 we simply need to compute the angular velocity times the time. For stage 3 the contribution is given by the rotation number found in Lemma 5. Montgomery [10] gave a splitting of the overall rotation of a rigid body into geometric phase and dynamic phase. Our formula is similar in spirit, with the added feature that there is no mod 2π for $\Delta\phi_{kick}$: We actually need to know how many somersaults occurred. We can prove the theorem by applying Lemma 5 to each stage of the dive. Stages 2 and 4 do not contribute because $\tau_2 = \tau_4 = 0$. In stages 1 and 5 the trajectory is at an equilibrium point on the **L**-sphere, so there is only a contribution to the dynamic phase. The essential terms come from stage 3, which is the twisting somersault stage without shape change. When computing S we need to choose a particular normalization of the integral which is different from [10, 9], and also different from [1]. Our normalization is such that when $J_x = J_y$ the amount of rotation obtained is the corresponding angle ϕ of the somersault, i.e., the rotation about the fixed axis **l** in space. This means that the correct solid angle for our purpose is such that when $J_x = J_y$ and the motion is approaching pure somersaulting the contribution from S approaches zero. Therefore, we should measure area relative to the equator on the sphere, as we did in Lemma 4. When $J_x = J_y$ we are simply measuring the area of a slice of the sphere bounded by the equator and the twisting somersault orbit. The orbit is contained in a plane, which is found by parallel translating the equatorial plane down by $l\sin(\chi + \mathcal{P})$, and then tilting it by the small angle $\mathcal{P}$; see Figure 1. The tilt can be ignored since it does not change the enclosed solid angle which thus is $2\pi\sin(\chi + \mathcal{P})$. In the general case where $J_x \neq J_y$ the area can be computed in terms of elliptic integrals as given in Lemma 4. Similarly, the period of the motion along $H = E_t$ can be computed either from explicit solutions of the Euler equations for $J_x = J_y$ leading to (21), or by elliptic integrals as in Lemma 5. ■

Now we have all the information needed to construct a twisting somersault. A result of the kick approximation is that we have $\tau_2 = \tau_4 = 0$, and if we further set $\tau_1 = \tau_5 = 0$, then there is no pure somersault either, which makes this the simplest twisting somersault. We call this dive the pure twisting somersault and take it as a first approximation to understanding the more complicated dives.

Corollary 7. *A pure twisting somersault with m somersaults and n twists is found for $\tau_1 = \tau_2 = \tau_4 = \tau_5 = 0$ and must satisfy*

$$2\pi m = \left(2lT_t\frac{E_t}{l^2} - S\right)n, \tag{22}$$

where both S and lT_t are functions of E_t/l^2 only (besides inertial parameters).

Proof. This is a simple consequence of the previous theorem by setting $\Delta\phi = 2\pi m$, $\tau_3 = nT_t$, and $\tau_1 = \tau_5 = 0$. ■

Remark 7.1. Solving (22) for m/n gives a rotation number of the Euler top, which characterizes the dynamics on some 2-tori of the superintegrable Euler top. This rotation number is equivalent up to unimodular transformations to that of Bates, Cushman, and Savev [1]; see Remark 5.2.

Remark 7.2. The number of somersaults per twists is m/n, and (22) determines $E_\mathfrak{t}/l^2$ (assuming the inertial parameters are given). Having $E_\mathfrak{t}/l^2$ determined in this way means one would need to find a shape change or kick which achieves that $E_\mathfrak{t}/l^2$, and large values of $E_\mathfrak{t}/l^2$ can be hard or impossible to achieve. For the one-arm kick-model discussed above the energy that is reached is given by

$$\frac{E_\mathfrak{t}}{l^2} = \frac{1}{2}\mathbf{L}_\mathfrak{s} R_x(\chi + \mathcal{P}) J R_x(-\chi - \mathcal{P})\mathbf{L}_\mathfrak{s}/l^2. \tag{23}$$

Remark 7.3. Given a particular shape change (say, in the kick approximation) the resulting $E_\mathfrak{t}/l^2$ will in general not result in a rational rotation number and hence not be a solution of (22). In this case the pure somersault of stage 1 and/or stage 5 needs to be used to achieve a solution of (19) instead.

Remark 7.4. The signs are chosen so that S is positive in the situation we consider. Thus the geometric phase lowers $\Delta\phi$ and can be thought of as an additional cost to total rotation that twisting adds on to somersaulting.

The total airborne time T_{air} has small variability for platform diving and is bounded above for springboard diving. A typical dive has $1.5 < T_{air} < 2.0$ seconds. After $E_\mathfrak{t}/l^2$ is determined by the choice of m/n the airborne time can be adjusted by changing l (within the physical possibilities) while keeping $E_\mathfrak{t}/l^2$ fixed. Imposing $T_{air} = \tau_1 + \tau_5 + \tau_3$ we obtain the following.

Corollary 8. *A twisting somersault with m somersaults and n twists in the kick-model must satisfy*

$$2\pi m + nS = T_{air}\frac{2E_\mathfrak{s}}{l} + 2nlT_\mathfrak{t}\frac{E_\mathfrak{t} - E_\mathfrak{s}}{l^2}, \tag{24}$$

where $T_{air} - \tau_3 = \tau_1 + \tau_5 \geq 0$.

4. The general twisting somersault. The kick-model gives a good understanding of the principal ingredients needed in a successful dive. In the full model the shape-changing times τ_2 and τ_4 need to be set to realistic values. We estimate that the full arm motion takes at least about 1/4 of a second. So instead of having a kick connecting $\mathbf{L}_\mathfrak{s}$ to $\mathbf{L}_\mathfrak{t}(0)$, a piece of trajectory from the time-dependent Euler equations needs to be inserted, which can be seen in Figure 3. The computation of the two dive segments from stages 2 and 4 has to be done numerically in general. Nevertheless, there is a beautiful generalization of Montgomery's formula [10] due to Cabrera [3], which holds in the nonrigid situation. In Cabrera's formula the geometric phase is still given by the solid angle enclosed by the trajectory; however, for the dynamic phase instead of simply $2ET$ we actually need to integrate $\mathbf{L}\cdot\boldsymbol{\Omega}$ from 0 to T. Now when the body is rigid we have $2E = \mathbf{L}\cdot\boldsymbol{\Omega} =$ const and Cabrera's formula reduces back to Montgomery's formula.

Theorem 9. *For the full model of a twisting somersault with n twists, the total amount of rotation $\Delta\Phi$ about the fixed angular momentum axis $\mathbf{l}$ is given by*

$$\Delta\phi = \Delta\phi_{kick} + \frac{2\bar{E}_2\tau_2}{l} + \frac{2\bar{E}_4\tau_4}{l} + S_-, \tag{25}$$

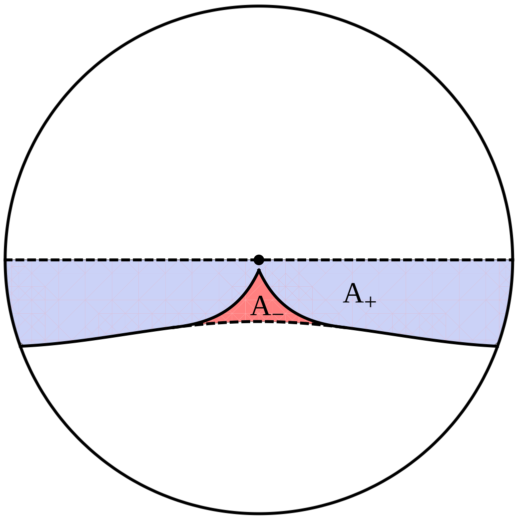


Figure 5. *Areas A_+ and A_- corresponding to solid angles $S_+ = A_+/l^2$ and $S_- = A_-/l^2$. The geometric phase correction due to the shape change is given by S_-.*

where S_- is the solid angle of the triangular area on the $\mathbf{L}$-sphere enclosed by the trajectories of the shape-changing stages 2 and 4 and part of the trajectory of stage 3; see Figure 5. The average energies along the transition segments are given by

$$\bar{E}_i = \frac{1}{2\tau_i}\int_0^{\tau_i} \mathbf{L}\cdot\mathbf{\Omega}\,\mathrm{d}t, \quad i = 2, 4. \tag{26}$$

Proof. This is a straightforward application of Cabrera's formula. For stages 1, 3, and 5, where there is no shape change the previous formula is obtained. For stages 2 and 4 the integral of $\mathbf{L}\cdot\mathbf{\Omega}$ along the numerically computed trajectory with time-dependent shape is computed to give the average energy during the shape change. ∎

Remark 9.1. This quantifies the error that occurs with the kick-model. The geometric phase is corrected by the solid angle S_- of a small triangle; see Figure 5. The dynamic phase is corrected by adding terms proportional to τ_2 and τ_4. Note that if we keep the total time $\tau_2 + \tau_3 + \tau_4$ constant, then we can think of the shape-changing times τ_2 and τ_4 from the full model as being part of the twisting somersault time τ_3 of the kick-model. The difference is $2\big((\bar{E}_2 - E_t)\tau_2 + (\bar{E}_4 - E_t)\tau_4\big)/l$, and since both $\bar{E}_2$ and $\bar{E}_4 < E_t$, the dynamics phase in the full model is slightly smaller than in the kick-model.

Remark 9.2. As E_t is found using the endpoint of stage 2 it can only be calculated numerically now.

The final step is to use the above results to find parameters that will achieve m somersaults and n twists, where typically m is a half-integer and n an integer.

Corollary 10. *A twisting somersault with m somersaults and n twists satisfies*

$$2\pi m + nS - S_- = T_{air}\frac{2E_s}{l} + 2\tau_2\frac{\bar{E}_2 - E_s}{l} + 2\tau_4\frac{\bar{E}_4 - E_s}{l} + 2\tau_3\frac{E_t - E_s}{l}, \tag{27}$$

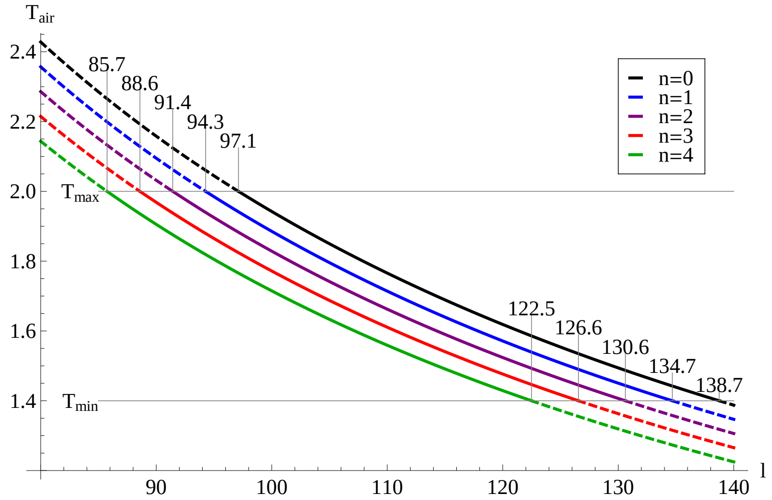


Figure 6. *The relationship between airborne time T_{air} and angular momentum l when $\tau_2 = \tau_4 = 1/4$ is used in Corollary 10. The result is for the case of $m = 1.5$ somersaults with different numbers of n twists. The maximum number of twists is $n = 4$ since we need $T_{air} - \tau_2 - \tau_3 - \tau_4 = \tau_1 + \tau_5 \geq 0$.*

where $T_{air} - \tau_2 - \tau_3 - \tau_4 = \tau_1 + \tau_5 \geq 0$.

Remark 10.1. Even though $\bar{E}_2, \bar{E}_4, E_t$, and S_- have to be computed numerically in this formula, the geometric interpretation is as clear as before: The geometric phase is given by the area terms nS and S_-.

In the absence of explicit solutions for the shape-changing stages 2 and 4, we have numerically evaluated the corresponding integrals and compared the predictions of the theory to a full numerical simulation. The results for a particular case and parameter scan are shown in Figures 3 and 6, respectively, and the agreement between theory and numerical simulation is extremely good. Fixing the shape change and the time it takes determines E_t/l^2, so the essential parameters to be adjusted by the athlete are the angular momentum l and airborne time T_{air} (which are directly related to the initial angular and vertical velocities at take-off). Our result shows that these two parameters are related in a precise way given in Corollary 10. At first it may seem counterintuitive that a twisting somersault with more twists (and same number of somersaults) requires less angular momentum when the airborne time is the same, as shown in Figure 6 for $m = 3/2$ and $n = 0, 1, 2, 3, 4$. The reason is that while twisting, the moments of inertia relevant for somersaulting are smaller than not twisting, since pure somersaults take layout position as shown in Figure 2a, and hence less overall time is necessary. In reality, the somersaulting phase is often done in pike or tuck position which significantly reduces the moment of inertia about the somersault axis, leading to the intuitive result that more twists require larger angular momentum when airborne time is the same.

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