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Indagationes Mathematicae ▮ (▮▮▮▮) ▮▮▮–▮▮▮

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The diver with a rotor

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Abstract

We present and analyse a simple model for the twisting somersault. The model consists of a rigid body with a rotor attached that can be switched on and off. This makes it simple enough to devise explicit analytical formulas whilst still maintaining sufficient complexity to preserve the shape-changing dynamics essential for twisting somersaults performed in springboard and platform diving. With "rotor on" and with "rotor off" the corresponding Euler-type equations can be solved and the essential quantities characterising the dynamics, such as the periods and rotation numbers, can be computed in terms of complete elliptic integrals. We arrive at explicit formulas for how to achieve a dive with m somersaults and n twists in a given total time. This can be thought of as a special case of a geometric phase formula due to Cabrera (2007).

Keywords: Rigid body dynamics; Gyroscope; Twisting somersault; Geometric phase

1. Introduction

The analysis of the twisting somersault poses an interesting problem in classical mechanics. How can a body take off in pure somersaulting motion, initiate twisting midflight, and then return to pure somersaulting motion for entry into the water? Generally this is not a problem of rigid body dynamics, but instead of either non-rigid body dynamics or the description of coupled rigid

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http://dx.doi.org/10.1016/j.indag.2016.04.003
0019-3577/

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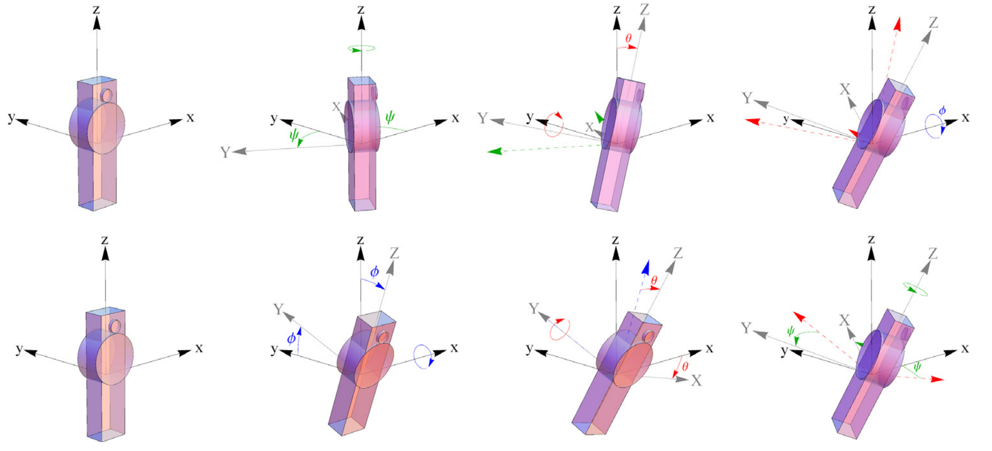

Fig. 1. A possible model of a rigid body (the box) and a disc that can be made to rotate about its symmetry axis. The box models the head, trunk, and legs of a human diver, while the disc models the arms, which can "rotate". In a robotic realisation the disc would actually rotate. The top line shows how the reference configuration is transformed to the final configuration by rotations about the initial axis. The bottom line shows how the same final configuration is reached by rotating about the intermediate axes in the reversed order.

bodies. Such a description of the twisting somersault was first proposed by [6,7] and has since been developed into a full-fledged analysis by Yeadon in a series of classical papers [14–18]. Here we are less ambitious in that we develop possibly the simplest model capable of exhibiting this kind of behaviour. The advantage of our model is that it is simple enough to be completely solved, thus allowing us to derive a precise equation that determines how exactly m somersaults and n twists can be performed in the total time T_{tot}, if at all. The model of the diver consists of a rigid body with a rotor attached. A rotation of the diver's arms is then simply modelled by switching the rotor on or off. The question we can answer is this: "When does the rotor need to be turned on to initiate twisting, and for how long should it stay on, off, and then on again to stop the twisting.?" From the dynamical systems point of view there are two autonomous systems ("rotor on" and "rotor off") that are switched between to achieve the desired trajectory. As such it is a discontinuous dynamical system whose solution is at least continuous. Despite its simplicity, the model appears to capture the essential features and even reasonable values of the parameters that are relevant in human springboard and platform diving. Whether we can learn something about human diving from this model – other than a rough idea of the fundamental principles – remains to be seen. However, we would like to propose that the simple device we are describing would make an interesting robot capable of performing twisting somersaults, potentially with many more twists than humanly possible.

2. Euler equations for a rigid body with a rotor

Let $\mathbf{l}$ be the constant angular momentum vector in a space fixed frame, and $\mathbf{L}$ the angular momentum vector in a reference frame moving with the body. Let R be the rotation matrix that transforms from one frame into the other, so that $\mathbf{l} = R\mathbf{L}$. The equations of motion for a rigid body with a rotor attached are well known, see e.g. [13,10,5,8,9]. Following Yeadon [16] we use an adapted system of Euler angles $R = R_1(\phi)R_2(\theta)R_3(\psi)$ where R_i is a rotation that fixes the ith axis, ϕ is the somersault angles, θ the tilt angle, and ψ the twist angle, see Fig. 1. This is the

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Euler-angle convention typically used in aerospace engineering where the angles are referred to as pitch, yaw, and roll.

Theorem 1. *The equations of motion for a rigid body with a rotating disc attached are given by*

$$l\begin{pmatrix}\cos\theta\cos\psi\\ -\cos\theta\sin\psi\\ \sin\theta\end{pmatrix}-\begin{pmatrix}0\\ h\\ 0\end{pmatrix}=\begin{pmatrix}I_1&0&0\\0&I_2&0\\0&0&I_3\end{pmatrix}\begin{pmatrix}\cos\theta\cos\psi&\sin\psi&0\\ -\cos\theta\sin\psi&\cos\psi&0\\ \sin\theta&0&1\end{pmatrix}\begin{pmatrix}\dot\phi\\ \dot\theta\\ \dot\psi\end{pmatrix}. \quad (1)$$

Proof. We start with the general Euler equations as e.g. derived in [12],

$$\dot{\mathbf{L}}=\mathbf{L}\times\mathbf{\Omega},\quad \mathbf{\Omega}=I^{-1}(\mathbf{L}-\mathbf{A}), \quad (2)$$

where $\mathbf{L}$ is the angular momentum in a body frame, I the tensor of inertia, and $\mathbf{A}$ the internal angular momentum generated by the rotating disc, in the present case $\mathbf{A}=(0,h,0)^t$. Using the constancy of $\mathbf{l}$ we can write the equations of motion as

$$R^t\mathbf{l}=I\mathbf{\Omega}+\mathbf{A} \quad (3)$$

where $\mathbf{\Omega}$ is determined by R through $\mathbf{\Omega}\times\mathbf{v}=R^t\dot R\mathbf{v}$ for any vector $\mathbf{v}\in\mathbb{R}^3$.

By choice of the space fixed coordinate system we may assume that $\mathbf{l}=(l,0,0)^t$. This means that the space fixed coordinate system is oriented so that the 1-axis is pointing to the right, the 2-axis to the front, and the 3-axis upwards, see Fig. 1. With the above choice of Euler angles we can find $\mathbf{\Omega}$ as on the right hand side of (1) and similarly $\mathbf{L}=R^t\mathbf{l}$ on the left hand side of (1). □

When the rotor is off we have $h=0$ and hence the internal momentum $\mathbf{A}$ vanishes. In this case the classical Euler equations are recovered. When the rotor is on we have non-zero h and hence $\mathbf{A}$ is non-zero and constant, and the equations of motion are as given in Theorem 1.

Due to the circular symmetry of the disc the tensor of inertia I of the "diver" will be constant regardless of the disc rotating or not. This is the essential simplification that makes this model tractable. A general shape change instead induces a time-dependent angular momentum shift $\mathbf{A}$ *and* a time-dependent tensor of inertia I.

Consider the caricature of a diver by a rectangular box (the trunk with legs and head attached, all rigidly connected) with a disc attached as shown in Fig. 1. The reference configuration is such that the 1-axis is pointing to the side of the body, the 2-axis out of the chest of the body, and the 3-axis up towards the head. The disc is attached so that it can rotate about an axis through the chest. The idea is to use the rotating disc to model the rotational up/down motion of the arms (and the hip and legs to a lesser extent). This will generate an internal angular momentum about the 2-axis, so that when the disc is rotating we have $\mathbf{A}=(0,h,0)^t$.

We have $h=\omega_d I_d$ where ω_d is the angular velocity of the disc, and I_d is its moment of inertia for rotation about its symmetry axis. These parameters need to be chosen so that we have a rough correspondence to the arm throw that initiates and stops the twisting motion. We estimate that moving the arm from "up" to "down", that is through an angle π takes at least 0.25 seconds, so that $\omega_d\le 4\pi$. Modelling each arm by a solid cylinder gives a value of $I_d\approx 2$, roughly one for each arm. It seems plausible to think of the disc as modelling the simultaneous motion of both arms rotating in the same direction, remaining parallel. We note that these numbers are just ballpark figures. As compared to the moments of inertia for the whole body, which we take to be $(I_1,I_2,I_3)=(20,21,1)$ (with both arms up), the rule of thumb is that the moment of inertia of the disc is similar to the moment of inertia for pure twisting of the whole body. These

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figures are computed from the model used in [12]. Much of our detailed analysis is done for the symmetric case (20, 20, 1). In the general case we only consider $I_1 < I_2$, so that the initial somersaulting takes place about the unstable axis of the body. Another ballpark figure to keep in mind is $l \approx 2\pi \cdot 20$ for the angular momentum of the whole body, which corresponds to the angular momentum necessary to perform one full somersault in one second. The fact that the moments of inertia (20, 21, 1) correspond to a planar diver is not essential for our analysis, increasing the smallest moment of inertia slightly so that the triangle inequality holds would not result in any qualitative change.

Corollary 2. *The equations of motion can be written as*

$$\phi' = 1 + \delta \sin^2 \psi + \hat{\rho} \sec\theta \sin\psi \tag{4a}$$
$$\theta' = -\delta \cos\theta \cos\psi \sin\psi - \hat{\rho} \cos\psi \tag{4b}$$
$$\psi' = \gamma \sin\theta - \delta \sin\theta \sin^2 \psi - \hat{\rho} \tan\theta \sin\psi \tag{4c}$$

where the prime $'$ *denotes the derivative* $d/d\tau$ *with respect to the dimensionless time* $\tau = tl/I_1$ *and*

$$\delta = \frac{I_1}{I_2} - 1, \qquad \gamma = \frac{I_1}{I_3} - 1, \qquad \rho = \frac{h}{l}, \qquad \hat{\rho} = \rho(1+\delta) \tag{5}$$

are dimensionless parameters. The dimensionless energy is a constant of motion and is given by

$$E = \frac{1}{2}\left(1 + \gamma \sin^2\theta + \delta \cos^2\theta \sin^2\psi\right) + \frac{1}{2}\hat{\rho}(\rho + 2\cos\theta \sin\psi). \tag{6}$$

Proof. Taking the equations from Theorem 1, dividing by l, then scaling time by I_1/l non-dimensionalises the equations, and solving for the derivatives of the angles gives the equations with the dimensionless parameters σ, δ, and ρ as stated. The energy is given by $E = \frac{1}{2}(\mathbf{L} - \mathbf{A})^t I^{-1}(\mathbf{L} - \mathbf{A})$ where $\mathbf{L} = R^t \mathbf{l}$. The fact that it is a constant of motion can be shown by direct computation, or alternatively using the fact that it is the Hamiltonian of the flow with respect to the Poisson structure $\mathbf{L}\times$. Non-dimensionalisation and expressing this in Euler angles gives the result. □

Remark 2.1. The symmetric case is found for $\delta = 0$ and the case with the rotor fixed is $\rho = 0$. In all cases ϕ does not appear on the right hand side.

Remark 2.2. For $\rho = 0$ the pure somersaulting equilibrium (ignoring ϕ) is at $\theta = \psi = 0$, and the eigenvalues of the linearisation about this equilibrium are $\pm\sqrt{-\delta\gamma}$. In particular the somersault is unstable when $\delta < 0$. This corresponds to $I_3 < I_1 < I_2$, which will be our standard assumption in the general case.

Remark 2.3. The non-dimensionalisation measures time in units of the inverse angular frequency l/I_1 of the pure somersault. Hence in scaled time after time 2π, a full pure somersault is executed, corresponding to $\phi' = 1$ when $\rho = \delta = 0$. This is why in the following figures we often plot the period divided by 2π.

3. The symmetric case

The dive can be separated into rigid and non-rigid stages. In the rigid stage $\rho = 0$ and for a symmetric body $\delta = 0$. Therefore, the equations of motion become trivial with $\phi' = 1$, $\theta' = 0$, and $\psi' = \gamma \sin\theta$ (a constant). Even when $\rho \neq 0$ ("rotor on") the equations of motion can be expressed completely in terms of θ:

Corollary 3. *In the symmetric case $I_1 = I_2 \iff \delta = 0$; the equations of motion are*

$$\phi' = 1 + \rho \sec\theta \sin\psi \tag{7a}$$

$$\theta' = -\rho \cos\psi \tag{7b}$$

$$\psi' = \gamma \sin\theta - \rho \tan\theta \sin\psi \tag{7c}$$

where

$$\sin\psi = \frac{E - \frac{1}{2}(1+\rho^2)}{\rho\cos\theta} - \frac{\gamma}{2\rho}\sin\theta\tan\theta. \tag{7d}$$

Proof. Solving the energy equation with $\delta = 0$ for $\sin\psi$ gives the result. Thus (at the expense of a square root in the $\dot\theta$ equation) the angle ψ can be eliminated on the right hand side. Note that the first term on the right hand side vanishes when $\delta = 0$. □

The dive is divided into five stages; it starts with a rigid stage ("rotor off") in pure somersaulting motion where $\mathbf{L} = \mathbf{l} = (l, 0, 0)^t = const$. In this stage we have $\phi' = 1$ and $\theta = \psi = 0 = const$. The time in stage i is denoted by T_i or $\hat{T}_i$ for the dimensionless time in stage i, the amount of somersault by ϕ_i, and the amount of twist by ψ_i. In stage 1 (pure somersault) we have $\psi_1 = 0$, and $\phi_1 = \hat{T}_1$.

In the non-rigid stage 2 the rotor is switched on. When the rotor is switched on the trajectory starts at $\mathbf{L} = (l, 0, 0)^t$ and we let it run until it reaches the maximum possible value of tilt $\theta_{\max}$ along that orbit, because the twist in the next stage will then be fastest, since $\psi' = \gamma\sin\theta$. In the next two lemmas we are going to compute the amount of time T_2 to complete the 2nd stage of the dive, and the amount of somersaulting ϕ_2 that occurs during this time. Let us remark that ϕ_2 can be interpreted as one quarter of the rotation number of the integrable system rigid body with a rotor.

Lemma 4. *The maximal θ that can be reached with "rotor on" from $\mathbf{L} = (l, 0, 0)^t$ is given by*

$$\cos\theta_{\max} = \sqrt{\beta^2 + 1} - \beta, \quad \beta = \frac{\rho}{\gamma}. \tag{8}$$

The inverse relation is

$$\beta = \frac{s^2}{2\sqrt{1-s^2}}, \quad s = \sin\theta_{\max}. \tag{9}$$

The time T_2 to move with "rotor on" from the point $\mathbf{L} = (l, 0, 0)^t$ to the point $\mathbf{L} = l(0, \cos\theta_{\max}, -\sin\theta_{\max})^t$ is

$$\frac{l}{I_1} T_2 = \hat{T}_2 = \frac{2k}{s\gamma} K(k^2), \quad k^2 = \frac{1-s^2}{2-s^2}. \tag{10}$$

Proof. By discrete symmetry the maximum $\theta_{\max}$ occurs for $\psi = \pm\pi/2$. Then, from the energy equation in Corollary 3 we obtain $\sin^2\theta \pm 2\beta\cos\theta = 0$, and hence the result. Using this equation β can be eliminated in favour of the variable $s = \sin\theta_{\max}$.

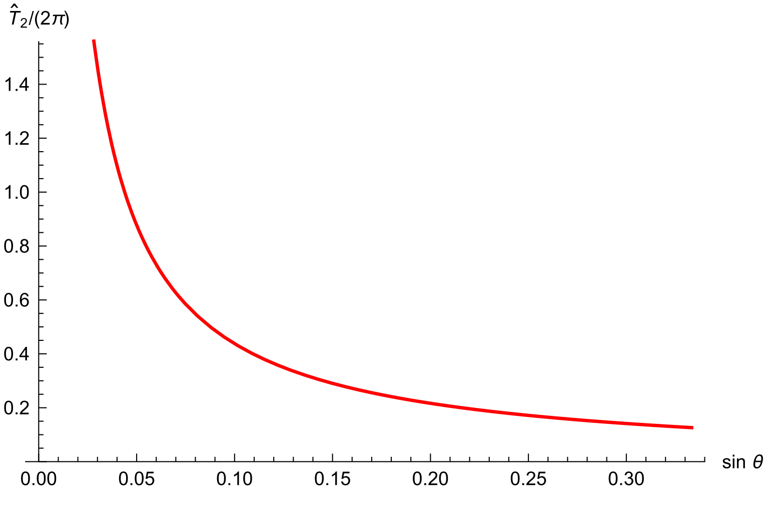

Fig. 2. Scaled time with rotor on $\hat{T}_2 = lT_2/I_1$ as a function of maximal tilt $s = \sin\theta_{\max}$, $\gamma = 19$, as given by (10).

Considering the θ' equation, separating the variables, and integrating from 0 to $\theta_{\max}$ gives

$$\int_0^{\theta_{\max}} \frac{2d\theta}{\gamma\sqrt{4\beta^2 - \sin^2\theta\tan^2\theta}} = \int d\tau = \hat{T}_2. \tag{11}$$

This is a complete elliptic integral of the first kind, which can be put into algebraic form with the substitution $z = \sin\theta$ so that the upper boundary is s. This can be expressed in terms of Legendre's K, see e.g. [3]. Un-scaling time gives the relation between T_2 and $\hat{T}_2$. □

Remark 4.1. The scaled time $\hat{T}_2$ for stage 2 (up to the overall factor $1/\gamma$) depends on the maximal tilt angle $s = \sin\theta_{\max}$ only, see Fig. 2. We use the term "maximal tilt angle" for s and $\theta_{\max}$ interchangeably, since they determine each other and for small tilt $s \approx \theta_{\max}$.

Lemma 5. *The amount of somersault that occurs with "rotor on" is given by*

$$\phi_2 = \frac{1}{s}\left(k(1+2\gamma^{-1})K(k^2) - (k^{-1} - k)\Pi(2 - k^{-2}, k^2)\right) \tag{12}$$

where $k^2 = \frac{1-s^2}{2-s^2}$ *as above.*

Proof. Using the equation for ϕ' from Corollary 3, transformed as $\frac{d\phi}{d\tau} = \frac{d\phi}{d\theta}\frac{d\theta}{d\tau}$ with the equation for θ' gives

$$\int_0^{\theta_{\max}} \frac{2 - \gamma\tan^2\theta}{\gamma\sqrt{4\beta^2 - \sin^2\theta\tan^2\theta}} d\theta = \int d\phi = \phi_2. \tag{13}$$

This is a complete elliptic integral of the third kind which can be put into algebraic form with the substitution $z = \sin\theta$ so that the upper boundary is s. Expressing it in terms of Legendre normal forms K and Π, see, e.g. [3], gives the result. □

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Remark 5.1. ϕ_2 is a function of the dimensionless parameters s and γ only. The combination $\hat{T}_2 - \phi_2$ is a function of s only.

With the analytical results for the amount of twist and somersault performed with rotor on, it is possible to derive conditions for a full twisting somersault in the symmetric case $I_1 = I_2$. The complete dive has 5 stages, pure somersaulting (stage 1) and "rotor-on" (stage 2) have already been described. The 3rd stage is rigid body motion with twist and somersault. Stage 4 is like stage 2 however with rotor on in the opposite direction, and stage 5 is pure somersaulting like stage 1. For the time T_i spent in stage i we therefore have $T_1 = T_5$ and $T_2 = T_4$.

Theorem 6. *To perform a dive with m somersaults and n twists in a total time $T_{tot} = 2T_2 + T_3 + 2T_1$, the master equation (see Fig. 4)*

$$\frac{l}{I_1} T_{tot} - 2m\pi = 2\frac{l}{I_1} T_2 - 2\phi_2 \tag{14}$$

needs to be satisfied, where the right hand side depends on the maximal tilt $s = \sin\theta_{\max}$ only. In addition, the condition that T_1 is non-negative (see Fig. 3)

$$2\frac{l}{I_1} T_1 = 2\pi m - 2\phi_2 - \frac{2\pi\left(n - \frac{1}{2}\right)}{\gamma \sin\theta_{\max}} \geq 0 \tag{15}$$

needs to be satisfied.

Proof. To achieve m somersaults we need $2m\pi = 2\phi_1 + 2\phi_2 + \phi_3$. To achieve n twists we need $2n\pi = 2\psi_1 + 2\psi_2 + \psi_3$. In the symmetric case and with "rotor off" (stage 1 and stage 3) the Euler equations simplify to $\phi' = 1$, see Corollary 3, so that $\phi_1 = \hat{T}_1$ and $\phi_3 = \hat{T}_3$. The other Euler equations are $\psi' = \gamma \sin\theta$ and $\theta' = 0$. In stage 1 we have $\theta = 0$, so $\psi_1 = 0$. In stage 3 we have $\theta = \theta_{\max} = const$, so $\psi_3 = \hat{T}_3 \gamma \sin\theta_{\max}$. For later reference it is useful to introduce the scaled period of the twist $\hat{P}_3 = 2\pi/(\gamma \sin\theta_{\max})$. Stages 2 and 4 ("rotor-on") together produce a half-twist, $\psi_2 = \psi_4 = \pi/2$, so the time T_3 has to be chosen so that $n - 1/2$ twists are generated in stage 3, so that overall n twists occur in stages 2 through 4:

$$\frac{l}{I_1} T_3 = \hat{T}_3 = \left(n - \tfrac{1}{2}\right) \hat{P}_3 = \frac{2\pi\left(n - \frac{1}{2}\right)}{\gamma \sin\theta_{\max}}. \tag{16}$$

For integer n, stages 2 and 4 are the same except for the sign of ρ. For half-integer n the same ρ is used and the final somersaulting in stage 5 occurs with $\mathbf{L} = l(-1, 0, 0)^t$. In either case we set the contribution from stage 1 and stage 5 to be equal.

The condition to have m somersaults gives

$$2\frac{l}{I_1} T_1 = 2\hat{T}_1 = 2\pi m - 2\phi_2 - \hat{T}_3. \tag{17}$$

The times $\hat{T}_2$ and $\hat{T}_3$ are non-negative for $n \geq 1/2$ and are determined by γ and the maximal tilt $s = \sin\theta_{\max}$. But for too large a value of n the formula for $\hat{T}_1$ gives a negative number, meaning that for those values of m, γ, and $\theta_{\max}$ the dive with this n is not possible.

Finally, the total time is simply the sum of the times of the stages. Eliminating T_1 and T_3 gives the final formula $\hat{T}_{tot} - 2m\pi$ as a function of s only, using the previous lemmas. □

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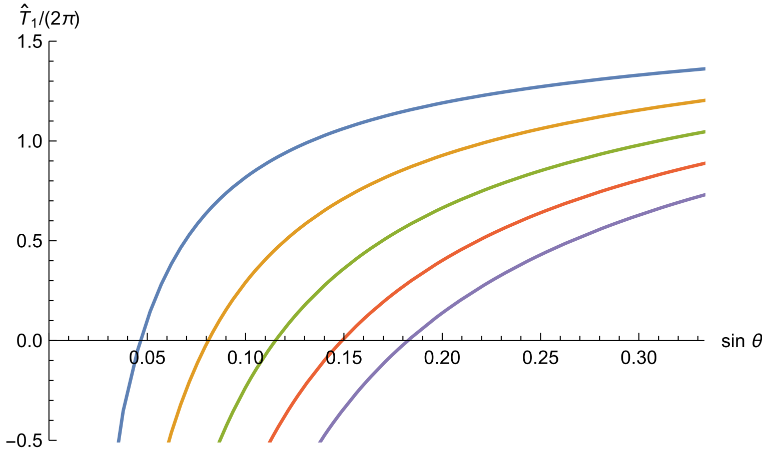

Fig. 3. Scaled somersaulting time $\hat{T}_1 = lT_1/I_1$ as a function of maximal tilt $s = \sin\theta_{\max}$ for $n = 1, 2, 3, 4, 5$ twists, $m = 3/2$ somersaults, $\gamma = 19$, as given by (15). The dive for given n is possible if $T_1 \geq 0$.

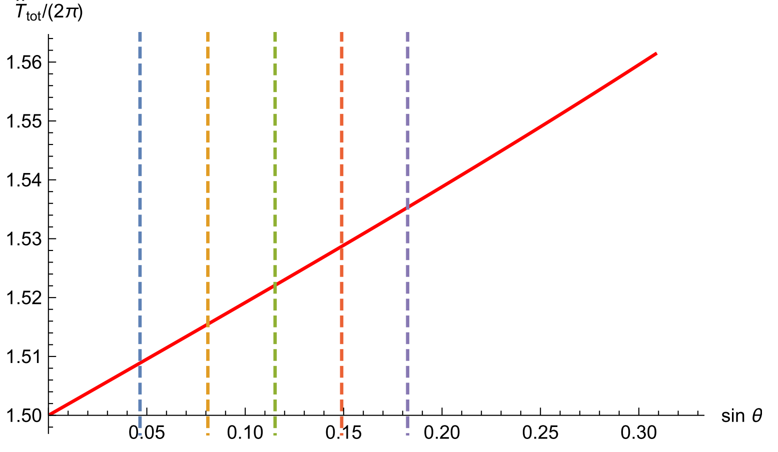

Fig. 4. Scaled total time $\hat{T}_{tot} = lT_{tot}/I_1$ as a function of the maximal tilt $s = \sin\theta_{\max}$ for $m = 3/2$ somersaults, as given by (14). The dashed lines indicate the minimal tilt needed for $n = 1, 2, 3, 4, 5$ twists.

When n is a half-integer entry into the water occurs facing the opposite way as compared to integer n. The half of the half-integer may be interpreted as the tennis racket half-twist [1] which occurs even when the rotor is not used, but instead the separatrix (or nearby) is traversed from one equilibrium on the **L**-sphere to the opposite one, e.g. see Fig. 6.

When designing a jump one fixes T_{tot}, n, and m, and tries to find a solution to the equations for which T_1 is non-negative. Solutions appear in one-parameter families, since the right hand side of (14) just depends on s, while on the left hand side only the product of l and T_{tot} is determined. So in the space of parameters h and l a curve is defined.

However, if one imagines that a particular solution is sought *after* takeoff, then l and T_{tot} are already fixed, and the only parameter still at disposal is the speed of rotation h, and hence s. The fact that the solutions appear in 1-parameter families in (h, l) space, see Fig. 4, is thus crucial to allow for corrections after takeoff.

The equations of the main theorem possess solutions for realistic values of the parameters (as discussed in the beginning, $\gamma \approx 19$, $T_{tot} \approx 1.5$, $h \leq 8\pi$, $l \leq 50\pi$) for $m < 3$ and $n \leq 4$. Thus a typical value of $\beta = h/(l\gamma)$ is ≈ 0.01, which gives a corresponding $s = \sin\theta_{\max} \approx 0.14$, and this is almost the tilt required to achieve $n = 4$ twists, see Fig. 4. When m is too big then by (14) the necessary l or T_{tot} (or both) will be too big. When n is too big then by (15) the minimal necessary tilt $s = \sin\theta_{\max}$ that will keep T_1 non-negative will be so large that it cannot be achieved by a human diver. A robotic model, however, could probably achieve the necessary values of h. The reason we cannot do more somersaults with this simple model is that we have not included the possibility of moving into pike or tuck position. This would decrease I_1 and hence allow more somersaults for the same value of angular momentum l, since the essential parameter is the combination $\hat{T}_{tot} = lT_{tot}/I_1$.

Another interesting observation is that if $\hat{T}_{tot}$ is sufficiently big, which in practical terms means that l is sufficiently big (as T_{tot} is essentially determined by the height of the platform), then after takeoff the diver can still decide how many twists to do by adjusting the tilt generated and the timing of the individual stages.

To get a better understanding of what the two main Eqs. (14) and (15) are saying and how they depend on the parameters, we now provide Taylor expansions valid for small maximal tilt angle s.

Lemma 7. *For small $s = \sin\theta_{\max}$ we have the following leading order behaviour:*

$$2\hat{T}_2 - 2\phi_2 \approx \sqrt{2}\left(2E\left(\tfrac{1}{2}\right) - K\left(\tfrac{1}{2}\right)\right)s + O(s^3) \tag{18a}$$

$$2\hat{T}_2 \approx \frac{2\sqrt{2}}{s\gamma}K\left(\tfrac{1}{2}\right) + O(s). \tag{18b}$$

Proof. Note that T_{tot} does not depend on γ, specifically the right hand side of (14) depends on s (and hence $\beta = \rho/\gamma = h/(l\gamma)$) only:

$$\hat{T}_2 - \phi_2 = \frac{1}{s}\left(-kK(k^2) + (k^{-1} + k)\Pi(2 - k^{-2}, k^2)\right). \tag{19}$$

Furthermore, this combination is regular at $s = 0$ using $\Pi(0, k) = K(k)$, and since k is an even function of s the whole expression has vanishing limit for $s \to 0$. The first s-derivative of the right hand side is $2(k^{-1} - k)(2E - K)$ and hence the result. By contrast, $\hat{T}_2$ has a pole at $s = 0$ and Taylor expansion of $s\hat{T}_2$ gives the result. □

Theorem 8. *The leading order behaviour of the total scaled time $\hat{T}_{tot} = lT_{tot}/I_1$ is determined by*

$$\frac{\hat{T}_{tot}}{2\pi} = m + As + O(s^3) \quad \text{where } A = \frac{1}{\sqrt{2}\pi}\left(2E\left(\tfrac{1}{2}\right) - K\left(\tfrac{1}{2}\right)\right) \approx 0.1907. \tag{20}$$

The minimal twist $\theta^*_{\max}$ that is necessary to perform m somersaults and n twists is approximately given by

$$\sin\theta^*_{\max} \approx \theta^*_{\max} \approx \frac{B + n}{m\gamma}, \quad \text{where } B = \frac{\sqrt{2}}{\pi}K\left(\tfrac{1}{2}\right) - \frac{1}{2} \approx 0.3346. \tag{21}$$

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Proof. Using Lemma 7 in the master equation for $\hat{T}_{tot}$ gives the first result. The minimal $\theta_{\max}$ is determined by the condition that T_1 be equal to zero. In the formula for $\hat{T}_1$ one contribution comes from the pole in Lemma 7, and the other from the pole in s in $\hat{T}_3$, see Theorem 6. Combining them the equation $T_1 = 0$ can be approximately solved to get the stated result. □

This shows that (as expected) more twists can be generated with more tilt, and the additional "cost" in tilt of one twist is approximately $1/(m\gamma)$. It should be noted that increasing the tilt $s = \sin\theta_{\max}$ also increases $\hat{T}_{tot}$, see Fig. 4, and hence for fixed T_{tot} it increases the necessary total angular momentum. The minimal total scaled time that is feasible is simply given by the number of somersaults m, and increasing the tilt s increases $\hat{T}_{tot}$ by sA.

The agreement of these approximate formulas with the exact formulas shown in Figs. 3 and 4 is very good in the relevant range of s.

4. The general case

When all three moments of inertia are distinct then the rigid twisting stage 3 does not have constant tilt θ, and complete elliptic integrals are needed to express the time T_3 and the somersault ϕ_3. The corresponding expressions were elementary in the symmetric case. Since θ is not constant, $\sin\theta$ can no longer be used as a parameter. Instead we will use $s_- = \sin\theta_{\min}$ or $s_+ = \sin\theta_{\max}$, which are determined by the extremal values of θ in the twisting stage 3. Thus from equating the energy in Corollary 2 for $\rho = 0$ at $\psi = 0$ and $\psi = \pi/2$ we find

$$s_+^2 + (1 - s_+^2)\nu = s_-^2. \tag{22}$$

We assume that the somersault axis is the middle principal axis so that $I_2 > I_1 > I_3$. This means that the somersault is unstable. As a result the dimensionless parameter $\delta < 0$.

Starting from Corollary 2 we find

Lemma 9. *The period $\hat{P}_3$ of the twisting motion and the change of somersault angle Φ_3 during such a period are given by*

$$\gamma \hat{P}_3 = \frac{8}{(s_+ + s_-)\sqrt{1-\nu}} K(k^2), \quad k = \frac{s_+ - s_-}{s_+ + s_-}, \ \nu = \delta/\gamma \tag{23a}$$

and

$$\Phi_3 - \hat{P}_3 = \frac{8}{(s_+ + s_-)\sqrt{1-\nu}} \Big(\Pi(n_-, k^2) - \Pi(n_+, k^2) \Big), \quad n_\pm = k\frac{1 \pm s_-}{1 \mp s_-}. \tag{23b}$$

Note that we distinguish the period $\hat{P}_3$ from the time $\hat{T}_3$ spent in stage 3; similarly for the somersault angle Φ_3 per period and somersault angle ϕ_3 acquired in stage 3.

In this description it is convenient to use both s_- and s_+, but of course either one could be eliminated with (22). Expressing everything in terms of s_- has the advantage that the limit $s_- \to 0$ corresponds to the approach of the separatrix of the pure somersault, while $s_- \to 1$ corresponds to the approach of the pure twisting motion.

When the moments of inertia are all distinct, the choice of Euler angles which we used earlier in the paper is natural because it has the physical interpretation of somersault, tilt, and twist. However, this gives a complicated result when the analogue of Corollary 3 is derived. The reason is that in order to solve the energy for $\sin\psi$ when $\delta \neq 0$ additional square roots are introduced. Instead a system of Euler angles needs to be used that has a rotation about the axis of the rotor

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last, say $R = R_1(\tilde{\phi})R_3(\tilde{\theta})R_2(\tilde{\psi})$. We will use this system of Euler angles to compute the time T_2 and the somersault ϕ_2 in stage 2.

Theorem 10. *An alternate form of the equations of motion for a rigid body with a rotating disc attached is given by*

$$l\begin{pmatrix}\cos\tilde{\theta}\cos\tilde{\psi}\\ -\sin\tilde{\theta}\\ \cos\tilde{\theta}\sin\tilde{\psi}\end{pmatrix} - \begin{pmatrix}0\\ h\\ 0\end{pmatrix} = \begin{pmatrix}I_1 & 0 & 0\\ 0 & I_2 & 0\\ 0 & 0 & I_3\end{pmatrix}\begin{pmatrix}\cos\tilde{\theta}\cos\tilde{\psi} & \sin\tilde{\psi} & 0\\ \sin\tilde{\theta} & 0 & 1\\ -\cos\tilde{\theta}\sin\tilde{\psi} & \cos\tilde{\psi} & 0\end{pmatrix}\begin{pmatrix}\dot{\tilde{\phi}}\\ \dot{\tilde{\theta}}\\ \dot{\tilde{\psi}}\end{pmatrix}. \tag{24}$$

The scaled equations of motion are

$$\dot{\tilde{\phi}} = 1 + \gamma\sin^2\tilde{\psi} \tag{25a}$$

$$\dot{\tilde{\theta}} = \gamma\cos\tilde{\theta}\sin\tilde{\psi}\cos\tilde{\psi} \tag{25b}$$

$$\dot{\tilde{\psi}} = -(1+\delta)\rho + \sin\tilde{\theta}(\gamma\sin^2\tilde{\psi} - \delta) \tag{25c}$$

with conserved energy

$$E = \frac{1}{2}\left((1+\gamma\sin^2\tilde{\psi})\cos^2\tilde{\theta} + (1+\delta)(\rho + \sin^2\tilde{\theta})\right). \tag{26}$$

In principle the computation of T_2 and ϕ_2 are similar to the symmetric case, but are a bit more elaborate. For stage 2 only $\theta_{\max}$ is defined, because stage 2 always starts with $\theta = 0$, but nevertheless we can use s_- from stage 3 as a parameter for stage 2.

Lemma 11. *The energy with "rotor on" is $E_2 = \frac{1}{2}(1 + \rho\hat{\rho})$ starting at $\mathbf{L} = (l, 0, 0)^t$, and the highest point on that orbit is $\mathbf{L} = (0, -\sin\tilde{\theta}_{\max}, \cos\tilde{\theta}_{\max})$ where*

$$-\sqrt{1 - s_+^2} = -\cos\theta_{\max} = \sin\tilde{\theta}_{\max} = \frac{\hat{\rho} - \sqrt{\hat{\rho}^2 + \gamma(\gamma-\delta)}}{\gamma - \delta}. \tag{27}$$

The time T_2 to move along this orbit segment is given by

$$\gamma\hat{T}_2 = \frac{1}{k\sqrt{s_-^2(1-\nu)+\nu}}K(k^2) \tag{28a}$$

where

$$k^2 = \frac{(s_-^{-2} - 1)(s_-^2(1-\nu)+\nu)}{(2 - s_-^2)(1-\nu)}, \quad \nu = \frac{\delta}{\gamma}. \tag{28b}$$

Proof. The point $\mathbf{L} = (l, 0, 0)^t$ corresponds to $\tilde{\theta} = \tilde{\psi} = 0$, while the point $\mathbf{L} = (0, *, *)^t$ implies $\tilde{\psi} = \pi/2$. Evaluating the energy at $(l, 0, 0)^t$ defines the energy E_2 with rotor-on, and evaluating $E_2 = E(\tilde{\psi} = \pi/2)$ defines the endpoint $\tilde{\theta}_{\max}$, which gives $\theta_{\max}$ and hence s_+. Then s_+ can be expressed in terms of s_- using (22).

Eliminating $\tilde{\psi}$ from the ODE for $\dot{\tilde{\theta}}$, separating the variables, and changing variables to $z = \sin\tilde{\theta}$ gives a first kind integral on the elliptic curve $w^2 = P(z) = z(z\delta + 2\hat{\rho})(z^2(\gamma-\delta) - 2z\hat{\rho} - \gamma)$. Using [3] this can be written in Legendre's normal form as stated. □

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Comparing the (scaled) energy $E_1 = 1/2$ of the pure somersaulting motion to that of the twisting motion $E_3 = E_1 + \frac{1}{2}\rho\hat{\rho}$, we see that the change in energy is proportional to ρ^2.

The angles ϕ and $\tilde{\phi}$ are related: upon a complete cycle in $\tilde{\theta}$ and $\tilde{\psi}$ (and hence $\mathbf{L}$) the overall advance in $\tilde{\phi}$ is the same as the overall advance in ϕ upon a complete cycle in θ and ψ (which is the same cycle in $\mathbf{L}$), but only modulo 2π. It turns out that in our case the total change of the two angles over a period in $\mathbf{L}$ differs by 2π, and hence $\phi_2 = \tilde{\phi}_2 - \pi/2$.

Lemma 12. *The change of angle $\tilde{\phi}_2$ in the general case satisfies*

$$\tilde{\phi}_2 - \hat{T}_2 = f_+ \Pi(n_+, k^2) + f_- \Pi(n_-, k^2) \tag{29a}$$

where k^2 is as in the previous lemma, $g = s_- \sqrt{(1-s_-^2)(2-s_-^2)(1-\nu)}$, and

$$\begin{aligned} n_\pm &= 1 - s_-^{-2} \pm s_-^{-2}\sqrt{(1-s_-^2)/(1-\nu)}, \\ f_\pm &= g\left(1 - s_-^2 \mp \sqrt{(1-s_-^2)(1-\nu)}\right). \end{aligned} \tag{29b}$$

Proof. The change in angle is given by a complete elliptic integral on the same elliptic curve $w^2 = P(z)$ as in Lemma 11, but with an additional rational function $R(z)$ obtained from $\frac{d\tilde{\phi}}{d\tau} = \frac{d\tilde{\phi}}{d\tilde{\theta}}\frac{d\tilde{\theta}}{d\tau}$ and hence

$$\tilde{\phi}_2 = \int R(z)\frac{dz}{w}, \quad R(z) = 1 + \delta - \frac{\hat{\rho} - \delta/2}{1+z} - \frac{\hat{\rho} + \delta/2}{1-z}. \tag{30}$$

Using [3] this can be written in terms of K and Π as stated. □

The conditions to perform a successful dive are as before, namely $T_{tot} = 2T_1 + 2T_2 + P_3(n - 1/2)$, $2m\pi = 2\phi_1 + 2\phi_2 + \Phi_3(n - 1/2)$ and $2n\pi = 2\psi_1 + 2\psi_2 + 2\pi(n - 1/2)$. The last equation is trivially satisfied with $\psi_1 = 0$, $\psi_2 = \pi/2$. This leads to our final result:

Theorem 13. *To perform a dive with m somersaults and n twists in a total time T_{tot}*

$$\hat{T}_{tot} - 2m\pi = 2(\hat{T}_2 - \phi_2) + (\hat{P}_3 - \Phi_3)\left(n - \tfrac{1}{2}\right) \tag{31}$$

and

$$2\hat{T}_1 = 2m\pi - 2\phi_2 - \Phi_3\left(n - \tfrac{1}{2}\right) \geq 0 \tag{32}$$

need to be satisfied.

Note that unlike in the symmetric case there is now a dependence of T_{tot} on n. What remains the same in the general case is that the right hand side of the equation for T_{tot} depends on the essential parameter s_- and the asymmetry parameter ν only. Thus as before s_- can be adjusted after takeoff to achieve the desired dive. The right hand side of the equation for T_1 in addition depends on γ, as it did in the symmetric case.

It turns out that the series expansion in the limit of small s_- has a rather limited radius of convergence, and so it is not useful to describe the range of s_- of interest to us. The reason behind this difficulty is that for $s_- \to 0$ the elliptic integrals have a logarithmic divergence.

Fig. 5 shows how the dives change when the asymmetry increases. The main observations are that with increasing asymmetry the necessary minimal tilt decreases, while the total necessary

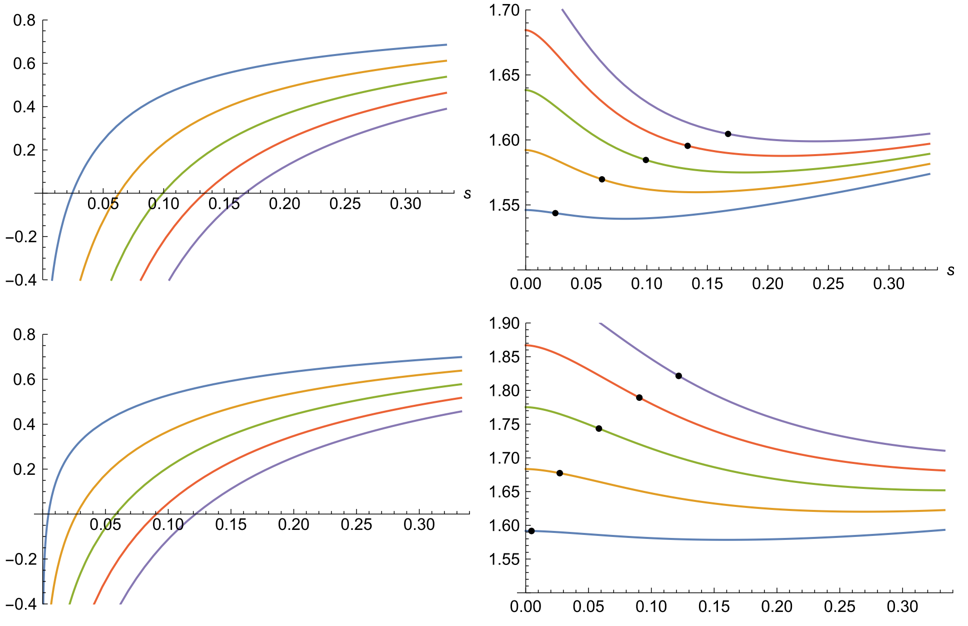

Fig. 5. Scaled stage one somersaulting time $\hat{T}_1/(2\pi)$ (left) and scaled total time $\hat{T}_{tot}/(2\pi)$ (right) as a function of $s_- = \sin\theta_{\min}$ for $n = 1, 2, 3, 4, 5$ in the general case for $m = 3/2$ somersault, $\gamma = 19$, for $\delta = -0.1$ (top row) and $\delta = -0.4$ (bottom row) as given by (32), (31). Dots mark the minimal s_- for which the dive is possible.

time increases. This can be understood by the presence of a separatrix in the asymmetric case. On the one hand the motion along the separatrix (even without the rotor) will produce an increase in tilt, in fact from s_- to s_+, as introduced above. On the other hand the motion near the separatrix will take longer when near s_-. In fact motion along the separatrix with rotor off takes an infinite amount of time, which is why the series expansions are not useful in this limit.

5. Geometric phase

The change in the somersault angle ϕ during a completion of a loop in $\mathbf{L}$ can be split into two contributions, a so called dynamic and a geometric phase. We start by discussing this in the simplest case of symmetric moments of inertia $I_1 = I_2$ and with rotor off, $\rho = 0$. In this case we can write

$$\Phi_3 = \frac{2E_3P_3}{l} - S_3 \tag{33}$$

where P_3 is the period of the twisting motion, E_3 the corresponding energy, and S_3 is the solid angle enclosed by the trajectory on the $\mathbf{L}$-sphere. For $I_1 = I_2$ the trajectory on the $\mathbf{L}$ sphere is a circle $\theta = const$, and the solid angle enclosed by this curve and the equator is $S_3 = 2\pi\sin\theta$ where $\theta = 0$ is the equator (zero solid angle), and $\theta = \pi/2$ is the pole (solid angle of half the sphere). The period of twist we found before, it is $P_3 = 2\pi I_1/(l\gamma\sin\theta)$, while $E_3 = l^2(1 + \gamma\sin^2\theta)/(2I_1)$. As noted before $\Phi_3 = lP_3/I_1$, and hence we have verified the above identity.

For general moments of inertia (33) still holds. Of course now the expressions for Φ_3, P_3 and S_3 are all complete elliptic integrals. Such a geometric phase formula with $\bmod 2\pi$ on the

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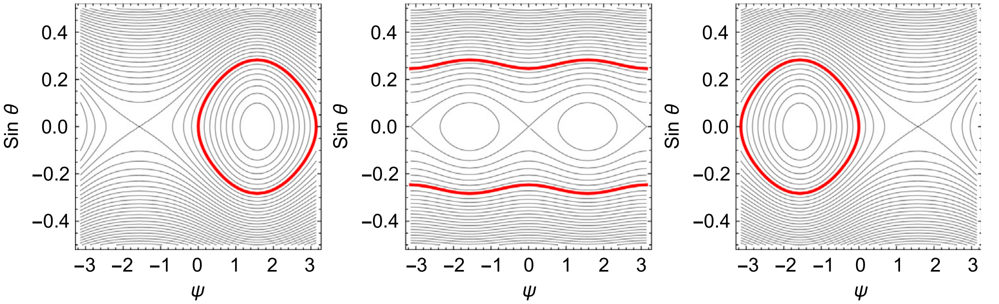

Fig. 6. Phase portraits in $(\psi, \sin\theta)$ for stages 2, 3, 4, corresponding to "rotor left", "rotor off", "rotor right". Parts of the three pieces of trajectories indicated in bold joined together give a dive as computed by the formulas in the previous section. Parameters are $\gamma = 19$, $\delta = -0.4$, $\rho = 1$.

right hand side was first derived by Montgomery [11]. For a particular choice of reduction the $\mathrm{mod} 2\pi$ can be removed, and this has first been done in [2]. We consider a different reduction based on our particular choice of Euler angles instead of Poinsot's description of the rigid body as in [2]. Thus our Eq. (33) gives the correct somersault angle ϕ without $\mathrm{mod} 2\pi$, under the condition that S_3 is measured relative to the equator, as defined above. In the present application it is clearly important to remove the $\mathrm{mod} 2\pi$, as we would like to distinguish between, say, 1/2 and 3/2 somersaults.

Including the "rotor on" stage, but returning to the symmetric case, we now have a closed loop in $\mathbf{L}$ that consists of pieces from stages 2, 3, and 4. There is a generalisation of (33) due to Cabrera [4], which allows for a general shape change. Before we write down the corresponding expression consider the three phase portraits for stages 2, 3, 4 on the $\mathbf{L}$-sphere. Here we present the $\mathbf{L}$-sphere in spherical coordinates (ψ, θ). This coordinate system is singular for $\theta = \pm\pi/2$, but the motions we are interested in do not come close to this point. In Fig. 6 three phase portraits are shown with the trajectories indicated as thick lines, which are used to construct the closed loop.

In Cabrera's formula the geometric phase is still given by the solid angle of the area enclosed. What changes is the dynamics phase where the simple $2ET$ is replaced by $\int \mathbf{L} \cdot \boldsymbol{\Omega}\, \mathrm{d}t$. Hence Cabrera's formula [4] in our notation (and without $\mathrm{mod} 2\pi$) is

$$\Phi = \frac{1}{l} \int \mathbf{L} \cdot \boldsymbol{\Omega}\, \mathrm{d}t - S. \tag{34}$$

This formula holds for arbitrary shape changes as long as the angular momentum $\mathbf{l}$ is conserved. In particular it holds for arbitrary time-dependent $\mathbf{A}$ and also for time-dependent tensor of inertia I, which together can describe more realistic shape changes. In our particular case the rotating disc produces a constant $\mathbf{A}$. When the solid angle S enclosed by the trajectory is measured relative to the equator this gives the correct overall somersault angle without $\mathrm{mod} 2\pi$, as we now show. With the choice of Euler angles as described in Theorem 1 we have

$$\mathbf{L} \cdot \boldsymbol{\Omega} = l(\dot\phi + \dot\psi \sin\theta). \tag{35}$$

The solid angle integral can be thought of as $\oint p dq$ where here $q = \psi$ and $p = \sin\theta$, so that the second term will be cancelled by the solid angle S (with appropriate sign and orientation), and the remaining integral gives the change in ϕ, as claimed. When $h = 0$ (rotor off) then it is easy to see that $\mathbf{L} \cdot \boldsymbol{\Omega} = 2E$ and the original geometric phase formula for the rigid body (33) is

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recovered. Again, for our particular choice of angle we measure the area relative to the equator, which allows us to remove the $\mod 2\pi$. The beauty of the resulting formulas is that they give us a good intuition of what the change in the somersault angle is going to be without actually computing it. Instead, we simply need to know the corresponding times and energies that give the dynamic phase, and the area enclosed by the equator and the curve on the **L**-sphere. Nothing will change in the description of the dive as presented in the previous sections, it is merely the interpretation of the answer that changes.

Acknowledgements

This research was partially supported by ARC Linkage grant LP100200245 with the New South Wales Institute of Sports.

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