Modeling in Calculus I

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Scenario 1: Combined Arms Live Fire

While participating in Cadet Field Training (CFT) in the summer of '99, you and your company are sent to Ft. Knox to conduct mounted maneuver training. You are placed in charge of the air strike portion of the Combined Arms Live Fire Exercise (CALFEX), involving Air Force A10s and Army Attack Helicopters. Your mission is to ensure the Air Force A10s and Army Attack Helicopters deliver their ordnance at the appropriate time and place, and that the air units maintain a minimum safe separation distance of 2 nautical miles. At 1400 hours, the OIC provides you with the following situation:

- At 1430 hrs Air Force A10s will be <u>171</u> nautical miles from the target area approaching at <u>206</u> knots on a heading of <u>303</u> degrees.
- At 1430 hrs the Army Attack Helicopters are <u>88</u> nautical miles from the target area, approaching at <u>103</u> knots on a heading of <u>213</u> degrees.

The OIC wants to know how fast the distance between the two air units will be changing at 1430 hrs and if the aircraft units maintain a minimum safe distance throughout the mission.

Problem 1: Determine a reference system to examine the rates of the aircraft.

Solution 1:

Reference System: Positive is moving away from the target. The target is the origin. Time zero (t = 0) is now (1430) hrs.

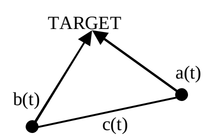

Figure 1: Sketch of the Air Mission

Problem 2: Define the known variables and constants.

Solution 2:

variables: $c(t)$ is the distance between the 2 air units at time t.(Hypotenuse)

$a(t)$ is the distance the army helicopters are from the target.

$$a(t)=-103t+88$$

$b(t)$ is the distance the air force A-10's are from the target at time t.

$$b(t)=-206t+171$$

$t(0)$ is 1430 hrs.

Army Attack Helicopters	Air Force A-10s
$a(0)$ = 88 nm	$b(0)$ = 171 nm
anglA = 303 degrees	angleB = 213 degrees
$a'(t)$ = 103 knots	$b'(0)$ = 206 knots

Problem 3: State some assumptions for this modeling problem.

Solution 3:

1. Both air units will maintain their given course and speed.
2. Both air units will maintain the same altitude.
3. The aircraft units maintain a tight formation and can be considered to be one point in space.

Problem 4: Determine the angle between the 2 aircraft units and the relationship between the units.

Solution 4:

The angle between the 2 aircraft units, angle = |anglA – anglB|. Therefore, angle = 90 degrees.

Therefore, Pythagorean Theorem applies!

Problem 5: Determine an expression relating all of the variables $a(t), b(t),$ and $c(t)$.

Solution 5:

$$c(t)^2 = a(t)^2 + b(t)^2$$

Problem 6: Implicitly differentiate $c(t)^2 = a(t)^2 + b(t)^2$ with respect to time.

Solution 6:

$$2 \cdot c(t) \cdot \frac{dc}{dt} = 2 \cdot a(t) \cdot \frac{da}{dt} + 2 \cdot b(t) \cdot \frac{db}{dt}$$

Problem 7: Simplify and solve for the change in distance between the two aircraft units.

Solution 7:

$$\frac{dc}{dt} = \frac{a(t) \cdot \frac{da}{dt} + b(t) \cdot \frac{db}{dt}}{c(t)}$$

Problem 8: Determine the change in distance between the 2 aircraft units at 1430 hrs (i.e., determine $c'(0)$).

Solution 8:

The distance between them at 1430 hours is decreasing and $\frac{dc}{dt} = 230.2994$ knots or nautical miles per hour.

Problem 9: In order to determine if the air units meet the mission, we need to determine how close the two air units will come to each other. Determine the minimum distance between the 2 planes. Do they violate the minimum safe separation distance of 2 nautical miles?

Solution 9:

We need find the minimum of $c(t) = \sqrt{a(t)^2 + b(t)^2}$. If we minimize what is underneath the radical $c(t)^2 = a(t)^2 + b(t)^2$, it follows that we will minimize the function $c(t)$.

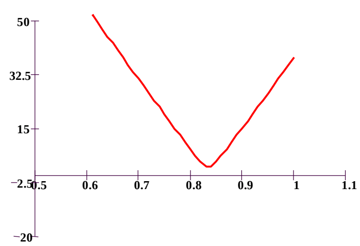

Figure 2: Plot of $c^2(t)$ vs. time.

Differentiate $c^2(t)$ implicitly: $\frac{dc}{dt} = a(t) \cdot \frac{da}{dt} + b(t) \cdot \frac{db}{dt}$ and solve or apply the MCad root finder:

guess: $t_1 = 0.82$ hours

$t = root(dc(t_1), t_1)$

$t = 0.8350$ hours

Conversion to clock time:

$0.8350\,\text{hr} \cdot \frac{60\,\text{min}}{1\,\text{hr}} = 50.0970\,\text{min}$ and $0.0970\,\text{min} \cdot \frac{60\,\text{sec}}{1\,\text{min}} = 5.8200\,\text{sec}$

Therefore, the air units will be closest to each other at 1520:05 hours.

We can now substitute $t = 0.8350$ back into Pythagorean's Theorem.

Thus, $MinC = \sqrt{a(0.8350)^2 + b(0.8350)^2}$ = 2.2361 nm.

Therefore, they will not violate the $2\,\text{nm}$ safety zone.

Problem 10: Are the assumptions you made reasonable?

Solution 10:

It is reasonable to assume the aircraft will fly straight and level at a constant velocity in a tight formation. This model is not based on many simplifying assumptions. The OIC should be very confident of the data and insights provided by this analysis. Initially, at 1430 hrs the aircraft units are closing in on each other at 230.3 knots per hour and during the exercise will maintain at least 2.236 nm distance from each other. If the aircraft deviate from the flight plans the OIC must be notified immediately.

Scenario 2: Tanks are not Discrete

Consider a tank battle between country X and Country Y. Analysts have reason to believe that if *x(t)* represents the number of tanks country X has after *t* hours of combat, and *y(t)* is the number of tanks Country Y has, then the differential equations governing a pure armor battle might take the form

$$\begin{cases} \dfrac{dx}{dt} = -A \cdot x \cdot y \\ \dfrac{dy}{dt} = -B \cdot x \cdot y \end{cases}$$

Problem 1a: Does this seem like a reasonable model? Explain.

Solution 1a:

A way to interpret this model is that we have a rate of change that is proportional to what is present, but the "constant" of proportionality depends on how many enemy are present: $\dfrac{dx}{dt} = -(A \cdot y) \cdot x$. Another way is to note that the factor $x \cdot y$ is a good measure of the interactions between the two populations: the higher each factor, the higher the product, and vice versa. A further consideration is that the rate of change is zero when there are no enemy present. While this may not accord with real life (see Problem 2), it is a reasonable model.

Problem 1b: What do the parameters A and B represent?

Solution 1b:

The parameter A is a measure of how fast country X loses tanks. As such, it is a combination of the probability of two opposing tanks meeting (and thus is dependent on time of day, weather, terrain, as well as tactical doctrine and target acquisition capabilities) as well as the probability of a "kill" once an encounter is made (and thus dependent on weapon characteristics, armor, etc.)

Problem 1c: What do you expect the long-term behavior of the system to be?

Solution 1c:

Given that there are no reinforcements, one expects that both forces will attrit (at rates which depend on the relative sizes of A and B) until one is down to zero, at which point the opposing force will remain at a constant strength from then on. The actual result (who "wins") would seem to depend on the relative sizes of A and B, but also on the initial conditions. Numerical experiments with different parameter values and different initial conditions can help here.

Problem 1d: Classify this system of differential equations.

Solution 1d:

This is a second-order, non-linear, homogeneous, autonomous (independent of time) system. As such, one expects not to be able to solve it analytically, however, as the next two parts show, this particular system can in fact be solved explicitly.

Problem 1e: Is $x(t)=\frac{1}{Bt},\quad y(t)=\frac{1}{At}$ a solution of this system?

Solution 1e:

We have $\frac{dx}{dt}=\frac{d}{dt}\left(\frac{1}{Bt}\right)=\frac{d}{dt}\left(\frac{1}{B}t^{-1}\right)=-\frac{1}{B}t^{-2}$ and similarly $\frac{dy}{dt}=-\frac{1}{A}t^{-2}$. Also, $-A\cdot x\cdot y=-A\cdot\left(\frac{1}{Bt}\right)\cdot\left(\frac{1}{At}\right)=-\frac{1}{Bt^2}$ and $-B\cdot x\cdot y=-\frac{1}{At^2}$. So this particular solution is verified.

Problem 1f: Is $x(t)=\frac{1}{B(t+C)},\quad y(t)=\frac{1}{A(t+C)}$ the general solution of this system? Why NOT?

Solution 1f:

We have $\frac{dx}{dt}=\frac{d}{dt}\left(\frac{1}{B(t+C)}\right)=\frac{d}{dt}\left(\frac{1}{B}(t+C)^{-1}\right)=-\frac{1}{B}(t+C)^{-2}$ and $\frac{dy}{dt}=-\frac{1}{A}(t+C)^{-2}$, while $-A\cdot x\cdot y=-A\cdot\left(\frac{1}{B(t+C)}\right)\cdot\left(\frac{1}{A(t+C)}\right)=-\frac{1}{B(t+C)^2}$ and

$-B \cdot x \cdot y = -\frac{1}{A(t+C)^2}$. Thus we have an infinite number of solutions, as C is an arbitrary constant. However, this is **NOT** a general solution, as we need two arbitrary constants to deal with all possible initial conditions. This solution will only work for initial conditions that have the ratio $\frac{x(0)}{y(0)} = \frac{A}{B}$.

Extra Credit: 1a. Show that $x(t) = \frac{D}{1-Fe^{-BDt}}$, $y(t) = \frac{\frac{B}{A}DFe^{-BDT}}{1-Fe^{-BDt}}$ is the general solution.

Solution Extra Credit 1a:

First note that $x \cdot y = \frac{D}{1-Fe^{-BDt}} \cdot \frac{\frac{B}{A}DFe^{-BDt}}{1-Fe^{-BDt}} = \frac{\frac{B}{A}D^2Fe^{-BDt}}{(1-Fe^{-BDt})^2}$. Also, $\frac{dx}{dt} = \frac{-BD^2Fe^{-BDt}}{(1-Fe^{-BDt})^2}$, and this is in fact $-A \cdot x \cdot y$. Furthermore,

$$\frac{dy}{dt} = \frac{(1-Fe^{-BDt})\left(-\frac{B^2D^2F}{A}e^{-BDt}\right) - \left(\frac{B}{A}DFe^{-BDt}\right)\left(BDFe^{-BDt}\right)}{(1-Fe^{-BDt})^2}$$

$$= \frac{-\frac{B^2D^2F}{A}e^{-BDt} + \frac{B^2D^2F^2}{A}e^{-2BDt} - \frac{B^2D^2F^2e^{-2BDt}}{A}}{(1-Fe^{-BDt})^2} = \frac{-\frac{B^2D^2F}{A}e^{-BDt}}{(1-Fe^{-BDt})^2}$$

and this is the same as $-B \cdot x \cdot y$. As there are two arbitrary constants in this formula, this is in fact the general solution to the system of equations.

Extra Credit: 1b. If A = 0.0009, B = 0.00075, $x(0) = 110$ and $y(0) = 100$, find the particular solution. What is the long-term result of an armor battle?

Solution Extra Credit 1b:

Substituting in $t = 0$, we get $120 = x(0) = \frac{D}{1-F}$ and

$100 = y(0) = \frac{BDF}{A(1-F)} = \frac{BF}{A}\left(\frac{D}{1-F}\right) = \frac{BF}{A}(120) = \frac{0.00075F}{0.0009}(120)$, and so

$F = \frac{110(0.0009)}{120(0.00075)} = 1.1$. Therefore $D = (1-F)120 = -0.1(120) = -12$. Thus the particular solution is $x(t) = \frac{-12}{1-1.1e^{0.009t}} = \frac{12}{1.1e^{0.009t}-1}$ and $y(t) = \frac{-11e^{0.009t}}{1-1.1e^{0.009t}} = \frac{11e^{0.009t}}{1.1e^{0.009t}-1}$. The result is that $x(t) \to 0$ and $y(t) \to 10$. So although Country Y started off with ten less tanks than

Country X, the greater survival rate meant that Country Y's armor force survived and Country X's didn't.

Problem 2: Suppose the differential equations are instead

$$\begin{cases} \dfrac{dx}{dt} = -0.0009 \cdot x \cdot y \\ \dfrac{dy}{dt} = -0.00065 \cdot x \cdot y - 0.01 \cdot y \end{cases}$$

What change in the situation does the new system above suggest?

Solution 2:

Notice that the parameter B has decreased from 0.00075 to 0.00065, but an additional term, $0.01 \cdot y$, has been subtracted. This would indicate that Country Y's tanks have become somewhat harder to kill (an improved design?) but they are lost at a steady 1% rate, regardless of the number of enemy present. One might speculate that the improved tank design is subject to breakdown, or perhaps the presence of a random element like a minefield is causing the steady decline in Country Y's tanks.

Problem 3: Suppose the differential equations are instead

$$\begin{cases} \dfrac{dx}{dt} = -0.0009 \cdot x \cdot y^{1.05} \\ \dfrac{dy}{dt} = -0.00065 \cdot x \cdot y - 0.01 \cdot y \end{cases}$$

What change in the situation does the new system above suggest?

Solution 3:

Now the exponent on the y term has slightly increased, so that large numbers of country Y's tanks will kill significantly more of Country X's tanks. Perhaps some new group-hunting doctrine or capability has been added to the Y side of the equation.

Problem 4: Suppose the differential equations are instead

$$\begin{cases} \dfrac{dx}{dt} = -0.0009 \cdot x \cdot y^{1.05} + 3 \\ \dfrac{dy}{dt} = -0.00065 \cdot x \cdot y - 0.01 \cdot y \end{cases}$$

What change in the situation does the new system above suggest?

Solution 4:

To combat this, Country X is now reinforcing the battle at a steady rate of three tanks per hour.

Problem 5: Suppose the differential equations are instead

$$\begin{cases} \dfrac{dx}{dt} = -0.0009 \cdot x \cdot y^{1.05} + 3 \\ \dfrac{dy}{dt} = -0.00065 \cdot x \cdot y - 0.01 \cdot y + 0.005 \cdot x \end{cases}$$

What change in the situation does the new system above suggest?

Solution 5:

Country Y is now reinforcing their forces also, but not at a steady rate. The rate of reinforcement depends on how many enemy tanks are still "alive".

Problem 6: Suppose the differential equations are instead

$$\begin{cases} \dfrac{dx}{dt} = -0.0009 \cdot x(t) \cdot y(t)^{1.05} + 3 \\ \dfrac{dy}{dt} = -0.00065 \cdot x(t) \cdot y(t) - 0.01 \cdot y(t) + 0.005 \cdot x(t-4) \end{cases}$$

What change in the situation does the new system above suggest?

Solution 6:

There is a four-hour (realistic?) time delay in the reinforcements for Country Y – information from the battle and logistics are delaying the reinforcement process.

Problem 7: How would you take into account friendly fire?

Solution 7:

Friendly fire would add probably negative x^2 and y^2 terms to the system, just as we use $x \cdot y$ terms to model interactions between forces, we use x^2 and y^2 terms to model interactions between friendly forces. One hopes the coefficients on these terms are very small.

Problem 8: What would the system look like if there is a three-way battle between Country X, Country Y, and Country Z? (This has occurred during World War II in China [the Japanese, the Chinese Communists, and the Kuomintang]; and in the former Yugoslavia during the breakup [the Serbs, the Bosnians, and the Croats].)

Solution 9:

One might expect, absent other factors as in #2-#7 above, to get a system of the form

$$\begin{cases} \dfrac{dx}{dt} = -A \cdot x \cdot y \cdot z \\ \dfrac{dy}{dt} = -B \cdot x \cdot y \cdot z \\ \dfrac{dz}{dt} = -C \cdot x \cdot y \cdot z \end{cases}$$

However, this is incorrect as it only seems to model three-way interactions, which should be rare. It also does not reduce to the two-way case when z = 0. A more realistic model might look like

$$\begin{cases} \dfrac{dx}{dt} = -A \cdot x \cdot y - D \cdot x \cdot z \\ \dfrac{dy}{dt} = -B \cdot x \cdot y - E \cdot y \cdot z \\ \dfrac{dz}{dt} = -C \cdot x \cdot z - F \cdot y \cdot z \end{cases}$$

where each parameter now measures interactions and kill rates for each possible two-way encounter. The complications introduced in #2-#7 could also be modeled.